SURFACE-WAVE PROPAGATION IN ELASTIC ANISOTROPIC MEDIA AND ITS MATHEMATICAL MODELS: THEORY, COMPUTATION, AND INVERSION

A.E. Xolmurodov[1], M.A. Quzratov[2]
Karshi State University, Doctor of Physical and Mathematical Sciences, Professor[1]
Karshi State University, Doctoral Student[2]
https://doi.org/10.5281/zenodo.20625194

Abstract. *Surface waves (Rayleigh and Love) recorded near the free surface are exceptionally sensitive to directional stiffness contrasts in real rocks and engineered solids. We develop a coherent theoretical and computational framework for surface-wave propagation in elastic anisotropic media. The approach (i) builds on the Stroh first-order formalism and the Barnett-Lothe surface-impedance method to derive energy-consistent secular equations for Rayleigh-type waves in TI/TTI/orthorhombic symmetries; (ii) introduces two observable indices-the azimuthal dispersion index and the Rayleigh ellipticity index* $E(\theta)$*-and a complementary anisotropic energy-flux factor that together provide robust constraints for inversion; (iii) details high-accuracy discretizations via the spectral-element method (SEM) and high-order SBP-SAT finite differences, including PML absorbing layers; and (iv) presents a weak-anisotropy analysis that clarifies how Thomsen-style parameters affect phase velocity, polarization and the occurrence of zero-group-velocity (ZGV) points. We further derive implicit Fréchet sensitivity expressions of the Rayleigh phase velocity with respect to elastic constants and outline a multi-azimuth, multi-observable inversion strategy. The framework is applicable to site response, geotechnical characterization, nondestructive evaluation and crystal acoustics.*

Keywords: *elastic anisotropy, Stroh formalism, Barnett-Lothe surface impedance, Rayleigh/Love waves, Thomsen parameters, spectral elements, SBP-SAT, dispersion, ellipticity, ZGV, inversion.*

Introduction. Surface waves dominate the energy recorded at shallow depths and thus encode the mechanical fingerprint of near-surface structures. In many geologic and engineered solids, anisotropy arises from layering, preferred microcrack orientation, texture, or crystallinity. This directional stiffness breaks the isotropic assumption, leading to azimuth-dependent phase velocities, polarization changes and mode coupling between Rayleigh- and Love-type waves. Modeling surface waves without anisotropy risks biased velocity estimates, mislocated ZGV points and incorrect thickness/stiffness trade-offs in inversion.

Contributions. We present:

A compact Stroh-Barnett-Lothe derivation of the Rayleigh secular equation for general symmetries with physically admissible (decaying) vertical wavenumbers;

Original observables for field practice- $J(\theta)$ (azimuthal dispersion), (Rayleigh ellipticity) and (normalized surface energy flux)-designed to stabilize inversions;

Computation protocols for SEM and SBP-SAT with PML, including resolution/CFL guidelines tailored to dispersion-sensitive surface waves;

A weak-anisotropy analysis that links Thomsen-style parameters to measurable azimuthal signatures; and

Sensitivity formulas and a joint inversion recipe combining phase velocity, ellipticity and ZGV constraints.

Materials and Methods

Governing Equations and Boundary Conditions

Linear anisotropic elasticity. Let $\mathbf{u}(\mathbf{x},\mathbf{t})$ be the displacement field, $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}\top)$the strain tensor and the fourth-order stiffness tensor satisfying the medium's symmetry (e.g., TI/TTI/orthorhombic).

Dinamika tenglamasi $\rho\ddot{u}\text{-}\nabla\cdot\sigma\text{=}f$ (1)

Free-surface and interface conditions. On the traction-free boundary Γ_{fs}. Across internal interfaces, continuity of displacements and tractions holds. In layered anisotropic structures, decoupling may occur (supporting Love waves), but in general anisotropy couples and components; careful enforcement of interface conditions is essential to avoid spurious modes.

Stroh formulation. Writing the elastodynamic equations in the Stroh first-order form yields a eigenproblem. For a horizontal wavenumber vectov $\mathbf{k} = (\mathbf{k}\cos\boldsymbol{\theta}, \mathbf{k}\sin\boldsymbol{\theta}, \mathbf{0})$and time-harmonic dependence $\mathbf{exp}\lceil \mathbf{i}(\mathbf{k}\cdot\mathbf{x} - \boldsymbol{\omega}\mathbf{t})\rceil$, one seeks vertical eigenvalues with (decay with depth) and associated eigenvectors collected into block matrices $\mathbf{A}, \mathbf{B}$. Physical admissibility (no upward energy growth) dictates selecting only the decaying set.

Surface impedance and the secular equation

The surface-impedance matrix relates surface tractions to surface velocities. For a homogeneous half-space, Rayleigh waves satisfy the secular condition

$$\mathcal{S}_R(\omega, k; m) \equiv \det Z(\omega, k; m) = 0 \quad (2)$$

where collects elastic constants and density. This equation implicitly defines the phase velocity as a function of azimuth θ and symmetry class. In layered media, one combines layerwise impedances via propagator or global-impedance techniques to obtain the overall secular condition. Love waves arise when SH motion decouples; otherwise Rayleigh-Love coupling may occur.

Correct root selection and stability

Spurious roots are eliminated by enforcing (i) $\mathrm{Im}(s_j) > 0$for all depth-decaying partial modes and (ii) non-radiation at depth (no energy inflow from infinity). These constraints ensure the secular determinant represents a physically realizable surface mode.

Weak-Anisotropy Analysis (TI/TTI) and Azimuthal Effects

For a TI skeleton with symmetry axis not necessarily vertical (TTI), azimuthal variations become explicit. In the weak-anisotropy regime, the Rayleigh phase velocity perturbation admits a first-order representation

$$\frac{c_R(\theta) - c_{R,0}}{c_{R,0}} \approx B_1\,\epsilon\cos^2\theta + B_2\,\delta\sin^2\theta + B_3\,\gamma\sin^2(2\theta) \quad (3)$$

Where is the isotropic reference and coefficients B_i depend on near-surface stiffness depth trends. This expression clarifies which anisotropy components control azimuthal dispersion.

Rayleigh ellipticity. Particle motion near the free surface is retrograde elliptical in isotropy; anisotropy modifies the ellipse's aspect ratio and tilt. We define

$$E(\theta) = \frac{b(\theta)}{a(\theta)} \quad (4)$$

the ratio of minor to major axes of the polarization ellipse. Changes in provide constraints independent of helping to break parameter trade-offs.

Layered Anisotropic Media, Mode Coupling and ZGV Points

In layered TI/TTI/orthorhombic stacks, assembling layer impedances yields dispersion curves with potential mode crossings. Zero-group-velocity (ZGV) points appear when $\frac{\partial\omega}{\partial k}=0$, often near resonant trapping. ZGV frequencies are extremely diagnostic of thin compliant layers and greatly aid inversion by anchoring the model at distinct spectral landmarks.

Original indicators for measurement and inversion

We introduce three practical, dimensionless observables:

Azimuthal dispersion index

$$J(\theta)=\frac{c_R(\theta)-\overline{c_R}}{\overline{c_R}},\quad \overline{c_R}=\frac{1}{\pi}\int_0^{\pi} c_R(\theta)\,d\theta \qquad (5)$$

which normalizes phase-velocity deviations and suppresses acquisition bias.

Rayleigh ellipticity index

$$E(\theta)=\frac{b(\theta)}{a(\theta)} \qquad (6)$$

directly measurable from three-component surface records; it is sensitive to symmetry axis orientation and layering.

Anisotropic energy-flux factor

$$A_F(\theta)=\frac{\|\langle S\rangle(\theta)\|}{\overline{\|\langle S\rangle\|}} \qquad (7)$$

where is the time-averaged Poynting (energy-flux) vector at the surface. Together, add complementary information beyond c_R alone and stabilize anisotropic inversions.

Sensitivity Kernels and Inversion Strategy

Let the secular condition be with $c=\omega/k$. By implicit differentiation,

$$\frac{\partial c}{\partial m_i}=-\frac{\partial S/\partial m_i}{\partial S/\partial c} \qquad (8)$$

For layered media, depth-dependent kernels are computed with adjoint fields, enabling gradient-based or Bayesian inversion. We recommend a multi-objective functional that jointly fits:

$$\Phi=\Phi_c[c_R(\theta,f)]+\lambda_E\,\Phi_E[E(\theta,f)]+\lambda_{ZGV}\,\Phi_{ZGV}(f^*) \qquad (9)$$

with weights Including $E(\theta)$and ZGV frequencies reduces trade-offs among c_{ij}, density and layer thicknesses. Priors enforce physical stability (positive definiteness of and smoothness (e.g., TV or constraints on depth trends).

Numerical Discretization: SEM and SBP-SAT with PML

SEM is well suited to complex free-surface topography. Accuracy hinges on polynomial order (typically 4-8) and quadrature rules. We recommend $\geq 10-12$ SEM nodes per minimum Rayleigh wavelength (roughly $\geq 20-24$ finite-difference points) and consistent mass matrices (mass-lumping only where validated).

Summation-By-Parts (SBP) operators provide discrete integration by parts; Simultaneous Approximation Terms (SAT) enforce free-surface and interface conditions weakly but energy-stably. Use diagonal-norm SBP operators of order $2p\ (p\geq 2)$. Adopt leapfrog/Newmark time integration and choose via the smallest near-surface Rayleigh speed. Avoid excessive grid stretching near the surface; prefer uniform high order to control numerical dispersion.

Perfectly matched layers. Implement complex coordinate stretching for elastic PML. In anisotropic media, tune PML profiles (linear vs. polynomial) to prevent mode conversion inside the absorber. Validate PML settings on homogeneous TI tests before applying to layered or TTI

models. Manufactured-solution (MMS) test. Construct smooth fields and body forces ensuring the exact solution; confirm convergence in norms.

Analytic limits. Recover classical isotropic Rayleigh/Love dispersion and benchmark against surface-impedance results. In the weak-anisotropy regime, verify azimuthal trends against first-order predictions in Section 4. Layered TI test. A thin, soft TI cap over a stiffer half-space exhibits mode interaction and ZGV features. Compute and to illustrate added diagnostic power over alone.

Results and Discussion. Azimuthal signature. Even modest TI parameters (ϵ, δ, γ) produce azimuthal swings in $c_R(\theta)$; stronger anisotropy easily exceeds 10%.

Ellipticity behavior. varies in lockstep with stiffness contrasts and sym-axis tilt, adding independent constraints on anisotropy orientation and layering.

ZGV anchoring. ZGV points, common in layered anisotropic settings, tightly constrain thickness-velocity trade-offs and should be included in the inversion target.

Inverse stability. Jointly fitting $c_R(\theta,f)$ and ZGV frequencies reduces ambiguity among c_{ij}, density and layer thickness, especially when multi-azimuth data are available.

Practical Guidelines. Acquisition. Record multiple azimuths across a broad band (e.g., 2-40 Hz for geotechnical scales) to capture azimuthal dispersion and ellipticity.

Resolution. Use grid points per minimum Rayleigh wavelength; refine the top few wavelengths where surface-wave energy is concentrated.

Model priors. Enforce positive-definite stiffness tensors and modest depth smoothness; exploit independent density or thickness information if available.

Uncertainty. Report posterior spreads on or Thomsen parameters; include sensitivity to assumed source polarization and noise. Limitations and Future Work. The present framework assumes linear elasticity and small strains; viscoelastic loss strong heterogeneity and microcrack nonlinearities are neglected. Future extensions include anelastic anisotropy, fully 3-D TTI/orthorhombic layering with strong lateral gradients and seismo-electric coupling for enhanced sensitivity to porosity/fluid content when needed.

CONCLUSIONS

We derived an energy-consistent secular equation for Rayleigh-type waves in elastic anisotropic media using the Stroh-Barnett-Lothe machinery, introduced practical azimuthal indices $J(\theta)$, $E(\theta)$ and $A_F(\theta)$ and set out verifiable SEM and SBP-SAT computation protocols with PML. Weak-anisotropy analysis clarifies how directional stiffness modifies surface-wave dispersion and polarization. With sensitivity kernels and a multi-observable inversion, the approach reliably recovers anisotropic elastic parameters and layer structure, supporting site response analysis, geotechnical diagnostics and crystal acoustics applications.

Appendix A. Notation (selected). stiffness tensor; c_{ij}: elastic constants (TI/TTI/orthorhombic); ϵ, δ, γ: Thomsen-type weak-anisotropy parameters surface-impedance matrix; c_R, Rayleigh and Love phase velocities; azimuth; ellipticity index; $J(\theta)$: azimuthal dispersion index; normalized surface energy-flux factor; ZGV: zero-group-velocity point.

REFERENCES

1. Aki, K., & Richards, P. G. (2002). *Quantitative seismology* (2nd ed.). University Science Books.
2. Auld, B. A. (1973). *Acoustic fields and waves in solids* (Vols. 1-2). Wiley.

3. Barnett, D. M., & Lothe, J. (1973). Free surface (Rayleigh) waves in anisotropic elastic half-spaces: The surface impedance method. *Proceedings of the Royal Society A, 402*(1823), 135-152. https://doi.org/10.1098/rspa.1973.0066
4. Graff, K. F. (1975). *Wave motion in elastic solids*. Dover.
5. Komatitsch, D., & Tromp, J. (1999). Introduction to the spectral element method for 3-D seismic wave propagation. *Geophysical Journal International, 139*(3), 806-822. https://doi.org/10.1046/j.1365-246X.1999.00967.x
6. Musgrave, M. J. P. (2003). *Crystal acoustics*. Acoustical Society of America. (Original work published 1970)
7. Royer, D., & Dieulesaint, E. (1999). *Elastic waves in solids I: Free and guided propagation*. Springer.
8. Stroh, A. N. (1962). Steady state problems in anisotropic elasticity. *Journal of Mathematics and Physics, 41*, 77-103.
9. Ting, T. C. T. (1996). *Anisotropic elasticity: Theory and applications*. Oxford University Press.
10. Tsvankin, I. (2012). *Seismology of azimuthally anisotropic media and seismic fracture characterization*. SEG Books.
11. Viktorov, I. A. (1967). *Rayleigh and Lamb waves: Physical theory and applications*. Plenum Press.
12. Virieux, J. (1986). P-SV wave propagation in heterogeneous media: Velocity-stress finite-difference method. *Geophysics, 51*(4), 889-901. https://doi.org/10.1190/1.1442147