IMPROVING THE METHODOLOGY FOR SOLVING EQUATIONS AND INEQUALITIES INVOLVING ABSOLUTE VALUE

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Abstract. *This article provides a scientific analysis of the theoretical and methodological foundations of teaching equations and inequalities involving absolute value, as well as the didactic potential of the analytical, graphical, geometric, and interval-partition methods used in solving them. It substantiates the need to interpret the concept of absolute value in the school algebra course not merely as an arithmetic operation or formal rule, but also as distance on the number line, a piecewise function, a graphical image, an algebraic model, and a tool for separating logical cases. The study discusses typical errors observed among students when solving absolute-value equations and inequalities, the causes of these errors, methodological approaches aimed at eliminating them, and didactic guidelines that support the gradual mastery of the topic. In addition, solutions to 10 sample complex problems on absolute-value equations and inequalities are presented with methodological comments; through these examples, the possibilities of developing students' analytical thinking, graphical imagination, reasoning on the number line, and competence in justifying solutions are revealed. The article is of theoretical and practical significance for mathematics teachers, methodologists, researchers in the field of pedagogy, and specialists interested in improving the methodology for effective teaching of the algebra course.*

Keywords: *absolute value, modulus, absolute-value equation, absolute-value inequality, interval method, graphical method, number line, methodology, algebra, mathematical thinking.*

Introduction

Solving equations and inequalities in the school mathematics course is one of the main directions in shaping students' algebraic culture. The concept of an equation is related to finding an unknown quantity, while an inequality is associated with determining the range of values of a variable; both develop skills in mathematical modeling, logical analysis, and functional thinking. In this process, equations and inequalities involving absolute value have special methodological significance, because they require students not to be limited to ordinary algebraic operations, but also to analyze the sign of an expression, divide the number line into intervals, visualize a graph, and verify the solution.

The concept of modulus is often presented as absolute value. However, from a methodological point of view, it is not sufficient to explain modulus only as "turning a value into a positive one." Above all, modulus expresses distance on the number line. For example, the following expression denotes the distance on the number line from point x to point a:

$$|x - a|$$

Therefore, when absolute-value equations and inequalities are explained through a distance-based interpretation, students understand the topic more deeply. For example, the equation

$$|x-3|=5$$

means that “the point x is located 5 units away from the point 3.” This interpretation immediately gives two solutions:

$$x=-2, x=8.$$

Such an approach develops students’ geometric imagination regarding the topic and makes it easier to understand subsequent complex problems.

The main difficulty in absolute-value equations and inequalities is that the sign of the expression inside the absolute value is unknown. Therefore, the student must consider each case separately. This shows the need to combine algorithmic and logical thinking when teaching topics involving absolute value. If a student attempts to “remove” the absolute value mechanically without taking into account the sign of the inner expression, an incorrect result is obtained.

The aim of the article is to analyze the methodology for solving equations and inequalities involving absolute value, to identify effective methods used in teaching the topic, and to develop methodological recommendations through complex examples.

Research Methods

The following methods were used in the study:

Theoretical analysis method. The algebraic, geometric, and functional meanings of the concept of absolute value were analyzed.

Comparison method. Definition-based, interval-partition, graphical, and distance-based interpretation methods for solving absolute-value equations and inequalities were compared.

Generalization method. General methodological conclusions were drawn on the basis of sample problems.

Didactic analysis method. The errors encountered by students and their causes were clarified.

Methodological design method. A step-by-step guide for teaching complex problems was developed.

Absolute value is studied on the basis of the following definition:

$$|a|=\begin{cases} a, & a\geq 0,\\ -a, & a<0. \end{cases}$$

This definition is the main theoretical basis for solving absolute-value equations and inequalities. If a variable occurs inside the absolute value, for example,

$$|f(x)|,$$

then the solution process depends on the sign of the function:

$$|f(x)|=\begin{cases} f(x), & f(x)\geq 0,\\ -f(x), & f(x)<0. \end{cases}$$

From a methodological perspective, this rule should not merely be memorized by students; rather, it should be explained through the number line, graphs, and real distance.

Results

The results of the study show that the following methodological system is effective in teaching equations and inequalities involving absolute value:

explaining simple absolute-value expressions as distance on the number line;

using basic equivalent transformations in equations and inequalities with one absolute value;

applying the interval-partition method in problems involving two or more absolute values;

predicting the number and location of solutions in advance using a graphical approach;

comparing the obtained solution with the given interval and verifying it without fail;

using absolute-value forms related to parameters, fractional-rational expressions, quadratic inequalities, and systems in complex tasks.

Below, 10 complex and methodologically useful problems selected for the article are presented as examples. In each problem, the mathematical expressions are given as separate formulas.

Example 1. An equation with two absolute values

Solve the equation:

$$|x-3|+|x+1|=8.$$

We find the points at which the expressions under the absolute values are equal to zero:

$$x-3=0 \Rightarrow x=3,$$
$$x+1=0 \Rightarrow x=-1.$$

Thus, the number line is divided into three intervals:

$(-\infty,-1),\ \ -1 \le x < 3,\ \ \ x \ge 3.$

On the first interval:

In this case:

$$|x-3|=3-x,$$
$$|x+1|=-x-1.$$

The equation:

$$(3-x)+(-x-1)=8.$$
$$2-2x=8.$$
$$-2x=6.$$
$$x=-3.$$

This value belongs to the first interval.

On the second interval:

$$-1 \le x < 3.$$

In this case:

$$|x-3|=3-x,$$
$$|x+1|=x+1.$$
$$(3-x)+(x+1)=8.$$
$$4=8.$$

This equality is false; therefore, there is no solution on this interval.

On the third interval:

$$x \ge 3.$$

In this case:

$$|x-3|=x-3,$$
$$|x+1|=x+1.$$
$$(x-3)+(x+1)=8.$$
$$2x-2=8.$$
$$2x=10.$$
$$x=5.$$

Answer:

This problem develops students' ability to divide the number line according to critical points and to open the absolute value correctly on each interval.

Example 2. An absolute-value inequality

Solve the inequality:

$$|2x - 5| < 7.$$

We use the basic property of an absolute-value inequality:

$$|A| < b(b > 0) \Leftrightarrow -b < A < b.$$

Thus:

$$-7 < 2x - 5 < 7.$$

We add 5 to all parts:

$$-2 < 2x < 12.$$

We divide by 2:

$$-1 < x < 6.$$

Answer:

$$x \in (-1,6).$$

Inequalities of the “absolute value is less than” type produce an inner interval around the center. It is recommended to illustrate this on the number line.

Example 3. An absolute-value inequality with outer intervals

Solve the inequality:

$$|3x + 2| \geq 10.$$

Basic rule:

$$|A| \geq b(b > 0) \Leftrightarrow A \leq -b\text{yoki}A \geq b.$$

Therefore:

$$3x + 2 \leq -10$$

or

$$3x + 2 \geq 10.$$

First inequality:

$$3x \leq -12,$$
$$x \leq -4.$$

Second inequality:

$$3x \geq 8,$$
$$x \geq \frac{8}{3}.$$

Answer: or

Inequalities of the “absolute value is greater than” type produce two outer intervals. Students often make the mistake of writing the intersection of intervals instead of their union in this case.

Example 4. An equation with three absolute values

Solve the equation:

$$|x - 2| + |x| + |x + 3| = 9.$$

We find the zeros of the equation, that is, the critical points:

$$x - 2 = 0 \Rightarrow x = 2,$$
$$x = 0,$$
$$x + 3 = 0 \Rightarrow x = -3.$$

Intervals:

On the first interval:

$$x \leftarrow 3.$$
$$|x-2| = 2-x,$$
$$|x| = -x,$$
$$|x+3| = -x-3.$$
$$(2-x)+(-x)+(-x-3) = 9.$$
$$-3x-1 = 9.$$
$$-3x = 10.$$
$$x = \frac{-10}{3}.$$

This value belongs to the first interval.

On the second interval:

$$-3 \leq x < 0.$$
$$|x-2| = 2-x,$$
$$|x| = -x,$$
$$|x+3| = x+3.$$
$$(2-x)+(-x)+(x+3) = 9.$$
$$5-x = 9.$$

Extraneous root; this value does not belong to the second interval.

On the third interval:

$$0 \leq x < 2.$$
$$|x-2| = 2-x,$$
$$|x| = x,$$
$$|x+3| = x+3.$$
$$(2-x)+x+(x+3) = 9.$$
$$x+5 = 9.$$

Extraneous root; this value does not belong to the third interval.

On the fourth interval:

$$x \geq 2.$$
$$|x-2| = x-2,$$
$$|x| = x,$$
$$|x+3| = x+3.$$
$$(x-2)+x+(x+3) = 9.$$
$$3x+1 = 9.$$
$$3x = 8.$$
$$x = \frac{8}{3}.$$

This value belongs to the fourth interval.

Answer:

$$x \in \left\{\frac{-10}{3}, \frac{8}{3}\right\}.$$

For equations involving three absolute values, it is useful to create a table: it should separately show how the absolute values are opened on each interval.

Example 5. A quadratic equation with absolute value

Solve the equation:

$$|x^2-5x+4| = 2.$$

Basic rule:

$$|A| = 2 \Leftrightarrow A = 2\text{yoki}A = -2.$$

Therefore, two equations are obtained:

$$x^2 - 5x + 4 = 2$$

and

$$x^2 - 5x + 4 = -2.$$

First equation:

$$x^2 - 5x + 2 = 0.$$

Discriminant:

$$D = (-5)^2 - 4 \cdot 1 \cdot 2 = 25 - 8 = 17.$$

$$x = \frac{5 \pm \sqrt{17}}{2}.$$

Second equation:

$$x^2 - 5x + 6 = 0.$$

$$(x - 2)(x - 3) = 0.$$

$$x = 2\text{yoki}x = 3.$$

Answer:

$$x \in \left\{2,3, \frac{5 - \sqrt{17}}{2}, \frac{5 + \sqrt{17}}{2}\right\}.$$

Methodological comment. In such problems, if a quadratic trinomial appears inside the absolute value, the absolute-value equation should first be split into two equations, and then each should be solved separately.

Example 6. A quadratic inequality with absolute value

Solve the inequality:

$$|x^2 - 4x - 5| \leq 7.$$

Basic rule:

$$|A| \leq b(b > 0) \Leftrightarrow -b \leq A \leq b.$$

Therefore:

$$-7 \leq x^2 - 4x - 5 \leq 7.$$

This double inequality is divided into two parts:

$$x^2 - 4x - 5 \geq -7$$

and

$$x^2 - 4x - 5 \leq 7.$$

First part:

$$x^2 - 4x + 2 \geq 0.$$

Roots:

$$x = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}.$$

Therefore:

$$x \leq 2 - \sqrt{2}.$$

or

$$x \geq 2 + \sqrt{2}.$$

Second part:

$$x^2 - 4x - 12 \leq 0.$$
$$(x-6)(x+2) \leq 0.$$
$$x \in [-2,6].$$

Now we take the intersection of both conditions:

$$-2 \leq x \leq 2 - \sqrt{2}.$$

or

$$2 + \sqrt{2} \leq x \leq 6.$$

Result:

$$x \in \left[-2, 2 - \sqrt{2}\right] \cup \left[2 + \sqrt{2}, 6\right].$$

Answer:

$$x \in \left[-2, 2 - \sqrt{2}\right] \cup \left[2 + \sqrt{2}, 6\right].$$

Methodological comment. In this problem, students learn to divide a double inequality into two separate inequalities and combine the results by taking their intersection.

Example 7. An equation with nested absolute values

Solve the equation:

$$\big||x-2| - 3\big| = 4.$$

We apply the rule with respect to the outer absolute value:

$$|x-2| - 3 = 4$$

or

$$|x-2| - 3 = -4.$$

First equation:

$$|x-2| = 7.$$
$$x - 2 = 7 \text{yoki} x - 2 = -7.$$
$$x = 9 \text{yoki} x = -5.$$

Second equation:

$$|x-2| = -1.$$

This equation has no solution, because an absolute value cannot be negative.

Answer:

$$x \in \{-5, 9\}.$$

In equations with nested absolute values, it is necessary to proceed step by step from the outer absolute value to the inner absolute value.

Example 8. An inequality with nested absolute values

Solve the inequality:

$$\big||x+1| - 2\big| \leq 3.$$

According to the rule:

$$-3 \leq |x+1| - 2 \leq 3.$$

We add 2 to all parts of the double inequality:

$$-1 \leq |x+1| \leq 5.$$

Since absolute value is always nonnegative,

$$|x+1| \geq 0.$$
$$|x+1| \leq 5.$$
$$-5 \leq x + 1 \leq 5.$$
$$-6 \leq x \leq 4.$$

Answer:

$$x \in [-6,4].$$

This problem teaches the practical use of the nonnegativity property of absolute value.

Example 9. An equation with absolute value and a parameter

Determine how many solutions the equation has:

$$|x-1| = a.$$

Here a is a parameter.

If

$$a < 0,$$

then the equation has no solution, because an absolute value cannot take a negative value.

If

$$a = 0,$$

then:

$$|x-1| = 0.$$
$$x - 1 = 0.$$
$$x = 1.$$

Thus, there is one solution.

If

$$a > 0,$$

then:

$$x - 1 = a$$

or

$$x - 1 = -a.$$
$$x = 1 + a$$

or

$$x = 1 - a.$$

Thus, there are two solutions.

Result:

$$\begin{cases} 0 \text{ ta yechim}, & a < 0, \\ 1 \text{ ta yechim}, & a = 0, \\ 2 \text{ ta yechim}, & a > 0. \end{cases}$$

The graphical method is also useful in absolute-value equations with parameters. Here, the number of intersection points of the graph y=|x-1| and the line y=a is analyzed.

Example 10. An inequality with two absolute values

Solve the inequality:

$$|x-4| - |x+2| \leq 3.$$

Critical points:

$$x = 4,$$
$$x = -2.$$

Intervals:

$$(-\infty, -2).$$
$$-2 \leq x < 4.$$
$$x \geq 4.$$

On the first interval:

$$x \leftarrow 2.$$
$$|x-4| = 4-x,$$
$$|x+2| = -x-2.$$
$$(4-x)-(-x-2) \leq 3.$$
$$6 \leq 3.$$

Thus, there is no solution on this interval.

On the second interval:

$$-2 \leq x < 4.$$
$$|x-4| = 4-x,$$
$$|x+2| = x+2.$$
$$(4-x)-(x+2) \leq 3.$$
$$2-2x \leq 3.$$
$$-2x \leq 1.$$
$$x \geq -\frac{1}{2}.$$

Intersection with the interval:

On the third interval:

$$x \geq 4.$$
$$|x-4| = x-4,$$
$$|x+2| = x+2.$$
$$(x-4)-(x+2) \leq 3.$$
$$-6 \leq 3.$$

This is true for all x≥4.

Thus:

General answer:

When combining solutions obtained on separate intervals, boundary points must be checked carefully. In systems, the intersection of solutions is taken, not their union. This problem strengthens students' correct understanding of logical connectives.

Discussion

The presented problems make it possible to identify several methodological directions in teaching equations and inequalities involving absolute value. The first direction is the correct application of the algebraic definition of absolute value. When opening an absolute value, the student must determine the sign of the inner expression. The second direction is the use of distance-based interpretation on the number line. This is especially effective when explaining simple equations and inequalities with one absolute value. The third direction is interval partitioning. This method is considered the most reliable in problems involving two or more absolute values. The fourth direction is graphical interpretation. The graphical method enables the student to predict the number of solutions in advance and to understand the shape of absolute-value functions.

The following didactic sequence is recommended when teaching the topic. First, the meaning of absolute value as distance on the number line is explained. Then simple equations, for example of the form |x-a|=r, are studied. After that, the inequalities |x-a|<r and |x-a|>r are interpreted through distance. At the next stage, equations and inequalities involving several absolute values are solved using the interval-partition method. At the final stage, absolute value is

applied in complex problems connected with quadratic functions, fractional-rational expressions, parameters, and systems.

One of the most common errors among students is solving an absolute-value equation in only one case. For example, in the equation |x-3|=5, writing only x-3=5 is incorrect, because the case x-3=-5 also exists. Another widespread error is writing the inequality |A|<b as A<b. The correct approach is -b<A<b. In addition, errors such as not finding the domain of definition in fractional absolute-value problems, choosing intervals incorrectly in quadratic inequalities, and taking the union instead of the intersection in systems are also observed.

From a methodological point of view, the following stages should be shown separately in each complex problem: domain of definition, critical points, intervals, opening the absolute value on each interval, solving the equation or inequality, checking whether the obtained solution corresponds to the interval, the general answer, and verification. This sequence develops students' orderly thinking and a culture of self-control.

Conclusion

Teaching equations and inequalities involving absolute value is one of the complex but very important sections of the algebra course. Through this topic, students master the definition of absolute value, its meaning as distance on the number line, its interpretation as a piecewise function, its graphical representation, and methods of analysis through intervals. For effective teaching of the topic, it is necessary not to be limited to memorizing formal rules, but to reveal their meaning from geometric, graphical, and analytical perspectives.

The analysis shows that the most effective methods for solving equations and inequalities involving absolute value are the following: the definition-based method, the distance method on the number line, the interval-partition method, the graphical method, equivalent transformations, and control of the domain of definition. When teaching complex problems, it is important to show each stage separately, verify the solutions, and write the answer clearly in interval form.

Thus, the topic of equations and inequalities involving absolute value has great methodological potential for developing students' logical, analytical, algorithmic, graphical, and geometric thinking. Teaching this topic deeply and systematically contributes to improving the quality of mathematics education.

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