SOME INEQUALITIES IN ONE TWO-BOUNDARY PROBLEM FOR STOCHASTIC PROCESSES WITH INDEPENDENT INCREMENTS

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https://doi.org/10.5281/zenodo.7783795

Abstract. *Two-sided estimates are found for the average value of the moment of the first exit from the interval of a homogeneous process with independent increments in the case when the process jumps are bounded from above and below.*

Keywords: *homogeneous process with independent increments, first exit time, ruin probability.*

Introduction. Let be $\xi(t),\ t\geq 0,\ \xi(0)=0,$ - a homogeneous random process with independent increments, whose sample functions are right-continuous. It is known that

$$Eexp\{\lambda\xi(t)\}=exp\{t\psi(\lambda)\},$$

$$\psi(\lambda)=\gamma\lambda+\frac{\sigma^2\lambda^2}{2}+\int_{-\infty}^{\infty}\left(e^{\lambda x}-1-\frac{\lambda x}{1+x^2}\right)dS(x), \qquad (1)$$

where γ and $\sigma>0$ are real numbers, the function $S(x)$ does not decrease on each of the intervals $(-\infty,\,0)$ and $(0,\,\infty)$,

$$\int_{|x|\leq 1} x^2 dS(x)<\infty,\ \ S(-\infty)=S(\infty)=0.$$

For arbitrary $a>0,\ b>0$ we introduce a random variable T, equal to the moment of the first exit of the process $\xi(t)$ from the interval $(-a,\,b)$:

$$T=T(a,\,b)=inf\,\{t\geq 0:\ \xi(t)\notin(-a,\,b)\}.$$

We suppose $T=\infty$, if $\xi(t)\in(-a,\,b)$ for all t. It is known ,that the random variable T is finite with probability one, if the distribution of the random variable $\xi(1)$ is not degenerate at zero, and $ET^k<\infty$ for all $k>0$.

The well-known problems of ruin, the theory of inventory storage, the theory of queuing systems, and a number of others lead to the study of the characteristics of random processes associated with the moment of the first exit from the interval.

Calculation in the exact form of the characteristics of random processes associated with the moment of exit from the interval is available only in some particular situations. Therefore, the main attention in the study of these characteristics is given to asymptotic approaches. The asymptotic analysis of the characteristics of two-boundary functionals is the subject of many works (some bibliographic information can be found, for example, in [1]).

Along with asymptotic formulas, the problem of obtaining two-sided estimates for the characteristics associated with the moment of the first exit of the random process from the interval is an urgent one. In the papers $[2]$, $[3]$ in the case of discrete time, under various restrictions on the distribution of the jump of a random walk, two-sided estimates were obtained for the probability of leaving the interval through the upper boundary. It is noted there that any asymptotic results inevitably contain remainder terms. **A** the assessment of the real value of these residuals requires additional consideration. Therefore, finding two-sided inequalities for the characteristics of boundary functionals is a natural addition to the available asymptotic results. The problem of finding the upper and lower bounds for the average value of the moment of the first exit from the random walk interval generated by the sums of independent identically distributed random variables is solved in $[4]$. A similar problem for homogeneous random processes with independent increments in various constraints on the distribution of the process was considered in [5]. This note is devoted to finding two-sided estimates for ET, that is, estimates for the average value of a random variable T in the case when the jumps of the process are bounded from above and below. Here we also assume $E\xi(1)=0$. The boundedness of the jumps of the process makes it possible to greatly simplify the calculation of the coefficients involved in the obtained two-sided inequalities. This is what distinguishes this work from [5].

To obtain two-sided bounds for ET , we need upper and lower bounds for the probabilities $P\left(\xi(T)\geq b\right)$ and $P\left(\xi(T)\leq -a\right)$, usually called ruin probabilities.

2. Inequalities for the probability of ruin. Let $E\xi(1)=0$ and

$$\int_{-\infty}^{c_-} dS(x)=0,\ \int_{c_+}^{\infty} dS(x)=0, \qquad (2)$$

that is, the jumps of process $\xi(t)$ are bounded from below and above, respectively, by the numbers c_- and c_+, $c_-<0$, $c_+>0$. It is obvious that there is a final mome $E\xi^k(1)$ for any k . According to the considerations from [5], the most interesting are the cases when $c_+\geq a+b$ or $c_-\leq -a-b$. Denote $\alpha(a,b)=P\left(\xi(T)\leq -a\right)$, $\beta(a,b)=P\left(\xi(T)\geq b\right)$ and we will find an estimate for them. By the Wald identity

$$E\xi(T)=E\xi(1)ET=0,$$

$$\begin{aligned}0=E\xi(T)&=E\left(\xi(T);\,\xi(T)\geq b\right)+E\left(\xi(T);\,\xi(T)\leq -a\right)=\\&=E\left(\xi(T)-b;\,\xi(T)\geq b\right)+bP\left(\xi(T)\geq b\right)+\\&+E\left(\xi(T)+a;\,\xi(T)\leq -a\right)-aP\left(\xi(T)\leq -a\right).\end{aligned}$$

Here and below, the designation of the form $E(Z;A)=EZI\{A\}$, is adopted, where $I\{A\}$ means the indicator of event A. Let's designate

$$\eta_+(b) = inf\left\{t \geq 0 : \xi(T) \geq b\right\},\ \eta_-(-a) = inf\left\{t \geq 0 : \xi(T) \leq -a\right\},$$

$$\chi_+(b) = \xi(\eta_+(b)) - b,\ \chi_-(-a) = \xi(\eta_-(-a)) + a,$$

that is $\eta_+(b)$ $(\eta_-(-a))$ - the moment of the first achievement of level b (-a) by process $\xi(t)$, and $\chi_+(b)$ $(\chi_-(-a))$ - jump over borders b (-a). Since $\alpha(a,b) + \beta(a,b) = 1$, we have

$$\beta(a,b) = \frac{a - E(\xi(T) - b;\, \xi(T) \geq b) - E(\xi(T) + a;\, \xi(T) \leq -a)}{a + b}. \tag{3}$$

It is clear that

$$E(\xi(T) - b;\, \xi(T) \geq b) = E(\chi_+(b);\, \xi(T) \geq b) = E\left(\chi_+(b) \cdot I\left\{\xi(T) \geq b\right\}\right),$$

and by the Cauchy-Bunyakovsky inequality for any $k \geq 0$

$$\begin{aligned} E\left((\xi(T) - b)^k;\, \xi(T) \geq b\right) &\leq \left(E\chi_+^{2k}(b)\right)^{\frac{1}{2}} \cdot \left(E\left(I\left\{\xi(T) \geq b\right\}\right)^2\right)^{\frac{1}{2}} = \\ &= \left(E\chi_+^{2k}(b)\right)^{\frac{1}{2}} \left(\beta(a,b)\right)^{\frac{1}{2}}. \end{aligned} \tag{4}$$

Likewise,

$$E\left(\left|\xi(T) + a\right|^k;\, \xi(T) \leq -a\right) \leq (E\,|\,\chi_-^{2k}(-a)\,|)^{\frac{1}{2}}(\alpha(a,b))^{\frac{1}{2}}. \tag{5}$$

Condition (2) immediately implies,that

$$E\chi_+^{2k}(b) \leq c_+^{2k},\ E\chi_-^{2k}(-a) \leq c_-^{2k} \tag{6}$$

and from (4), (5) at $k = 1$

$$\begin{aligned} &E\left((\xi(T) - b);\, \xi(T) \geq b\right) \leq c_+(\beta(a,b))^{\frac{1}{2}}, \\ &E\left(\left|\xi(T) + a\right|;\, \xi(T) \leq -a\right) \leq -c_-(\alpha(a,b))^{\frac{1}{2}}. \end{aligned} \tag{7}$$

To obtain a lower estimate for $\beta(a,b)$ we restrict ourselves to the lower zero estimate of the expression $E\left(\left|\xi(T) + a\right|;\, \xi(T) \leq -a\right)$, and when finding the upper estimate, we will replace $E(\xi(T) - b;\, \xi(T) \geq b)$ with zero. As a result, in accordance with (3), we have

$$\beta(a,b) \geq \frac{a - c_+(\beta(a,b))^{\frac{1}{2}}}{a + b},\ \alpha(a,b) \geq \frac{b + c_-(\alpha(a,b))^{\frac{1}{2}}}{a + b}. \tag{8}$$

The last relations can be considered as quadratic inequalities with respect to $\sqrt{\beta(a,b)}$ and $\sqrt{\alpha(a,b)}$. Solving these inequalities with respect to $\beta(a,b)$ and $\alpha(a,b)$ taking into account equality $\alpha(a,b) + \beta(a,b) = 1$, we obtain the following theorem.

Theorem 1. *Let the spectral function* $S(x)$ *in representation* (1) *satisfy condition* (2) *and* $E\xi(1) = 0$. *Then the following inequalities hold*

$$\frac{a}{a+b}-\frac{c_+\sqrt{c_+^2+4a(a+b)}-c_+^2}{2(a+b)^2}\le\beta(a,b)\le\frac{a}{a+b}-\frac{c_-\sqrt{c_-^2+4b(a+b)}+c_-^2}{2(a+b)^2},$$

$$\frac{b}{a+b}+\frac{c_-\sqrt{c_-^2+4b(a+b)}-c_-^2}{2(a+b)^2}\le\alpha(a,b)\le\frac{b}{a+b}+\frac{c_+\sqrt{c_+^2+4a(a+b)}-c_+^2}{2(a+b)^2}.$$

3. Inequalities for ET . Due to the Wald identity

$$E\xi^2(T)=\mu ET,\mu=\xi^2(1).$$

The justification of the last equality can be found, for example, in [6].

Further, it is easy to see that

$$E\xi^2(T)=a^2P(\xi(T)\le-a)+b^2P(\xi(T)\ge b)-2aE(\xi(T)+a;\,\xi(T)\le-a)+ \\ +2bE(\xi(T)-b;\,\xi(T)\ge b)+E\left((\xi(T)+a)^2;\,\xi(T)\le-a\right)+E\left((\xi(T)-b)^2;\,\xi(T)\ge b\right). \quad (9)$$

So, to estimate ET , estimates of each term on the right side of (9) are needed. Upper bounds for $E(\xi(T)-b;\,\xi(T)\ge b)$ and $E(|\xi(T)+a|;\,\xi(T)\le-a)$ are given in (7). And for the last two terms, the necessary upper bounds follow from (4) - (6) for κ=2:

$$E\left((\xi(T)-b)^2;\,\xi(T)\ge b\right)\le\left(E\chi_+^4(b)\right)^{\frac{1}{2}}\left(\beta(a,b)\right)^{\frac{1}{2}}\le c_+^2(\beta(a,b))^{\frac{1}{2}},$$

(10)

$$E\left((\xi(T)+a)^2;\,\xi(T)\le-a\right)\le c_-^2\left(\alpha(a,b)\right)^{\frac{1}{2}}.$$

Let us denote the right-hand sides of the inequalities from theorem 1 by

$$l_-=\frac{a}{a+b}-\frac{c_-\sqrt{c_-^2+4b(a+b)}+c_-^2}{2(a+b)^2},l_+=\frac{b}{a+b}+\frac{c_+\sqrt{c_+^2+4a(a+b)}-c_+^2}{2(a+b)^2}. \quad (11)$$

The upper estimate for ET follows from (7), (9), (10) .

To estimate ET from below, we restrict ourselves here to the following obvious relation

$$E\xi^2(T)=E\left(\xi^2(T);\,\xi(T)\le-a\right)+E\left(\xi^2(T);\,\xi(T)\ge b\right)\ge \\ \ge a^2P(\xi(T)\le-a)+b^2P(\xi(T)\ge b).$$

Applying lower bounds for $\alpha(a,b)$ and $\beta(a,b)$, Theorem 1 easily yields a lower bound for ET . Thus, the following theorem has been proved.

Theorem 2. Under the conditions of theorem 1 the inequalities

$$ET\le\mu^{-1}(a^2l_++b^2l_-++2ac_-\sqrt{l_+}+2bc_+\sqrt{l_-}+c_-^2\sqrt{l_+}+c_+^2\sqrt{l_-}),$$

$$ET\ge\mu^{-1}(a^2l_++b^2l_-),$$

l_+,l_- are defined in (11), $\mu=\xi^2(1)$.

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