TWO-DIMENSIONAL LEFT (RIGHT) UNITAL ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELDS AND R

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https://doi.org/10.5281/zenodo.10402063

Abstract. *The principal building blocks of our descriptions are derived from [1, 4] as the authors have presented complete lists of isomorphism classes of all two-dimensional algebras over algebraically closed fields and* $\mathbb{R}$*, providing the lists of canonical representatives of their structure constant's matrices.*

Keywords: *the latest lists of all unital associative algebras in dimension two, three, four.*

1. Introduction

The principal building blocks of our descriptions are derived from [1, 4] as the authors have presented complete lists of isomorphism classes of all two-dimensional algebras over algebraically closed fields and $\mathbb{R}$, providing the lists of canonical representatives of their structure constant's matrices. We will speak of a unity element and unital algebras rather than identity element and algebras with identity. The latest lists of all unital associative algebras in dimension two, three, four, and five are available in [10], [2], [6] and [9], respectively. The lists of all associative algebras (both unital and non-unital) in dimension two and three are presented in [5, 11]. In this paper we describe the isomorphism classes of two-dimensional left(right) unital algebras over arbitrary algebraically closed field and $\mathbb{R}$. Our approach is totally different than that of [2, 5, 6, 9, 10, 11].

We consider left(right) unital algebras over algebraically closed fields of characteristic not $2,3$, characteristic 2, characteristic 3 and over $\mathbb{R}$ separately according to classification results of [1, 4]. To the best knowledge of authors the descriptions of left(right) unital two-dimensional algebras over algebraically closed fields and $\mathbb{R}$ have not been given yet.

2. Preliminaries

Let $\mathbb{F}$ be any field, $A\otimes B$ stand for the Kronecker product consisting of blocks $(a_{ij}B)$, where $A=(a_{ij})$, B are matrices over $\mathbb{F}$.

Let $(\mathbb{A},\cdot)$ be m-dimensional algebra over $\mathbb{F}$ and $e=(e^1,e^2,...,e^m)$ its basis. Then the bilinear map is represented by a matrix $A=(A_{ij}^k)\in M(m\times m^2;\mathbb{F})$ as follows

$$\mathbf{u}\cdot\mathbf{v}=eA(u\otimes v),$$

for $\mathbf{u}=eu,\mathbf{v}=ev,$ where $u=(u_1,u_2,...,u_m)^T$, $v=(v_1,v_2,...,v_m)^T$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively. The matrix $A\in M(m\times m^2;\mathbb{F})$ defined above is called the matrix of structural constants (MSC) of $\mathbb{A}$ with respect to the basis e. Further we assume that a basis e is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC A (see [3]).

If $e'=(e'^1,e'^2,...,e'^m)$ is another basis of $\mathbb{A}$, $e'g=e$ with $g\in G=GL(m;\mathbb{F})$, and A' is MSC of $\mathbb{A}$ with respect to e' then it is known that

$$A'=gA(g^{-1})^{\otimes 2} \qquad (1)$$

is valid. Thus, the isomorphism of algebras $\mathbb{A}$ and $\mathbb{B}$ over $\mathbb{F}$ can be given in terms of MSC as follows.

Definition 2.1 *Two* m*-dimensional algebras* $\mathbb{A}$, $\mathbb{B}$ *over* $\mathbb{F}$, *given by their matrices of structure constants* A, B, *are said to be isomorphic if* $B = gA(g^{-1})^{\otimes 2}$ *holds true for some* $g \in GL(m;\mathbb{F})$.

Definition 2.2 *An element* $\mathbf{1_L}$ ($\mathbf{1_R}$) *of an algebra* $\mathbb{A}$ *is called a left (respectively, right) unit if* $\mathbf{1_L} \cdot \mathbf{u} = \mathbf{u}$ *(respectively,* $\mathbf{u} \cdot \mathbf{1_R} = \mathbf{u}$*) for all* $\mathbf{u} \in \mathbb{A}$. *An algebra with the left(right) unit element is said to be left(right) unital algebra, respectively.*

Definition 2.3 *An element* $\mathbf{1} \in \mathbb{A}$ *is said to be an unit element if* $\mathbf{1} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{1} = \mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{A}$. *In this case the algebra* $\mathbb{A}$ *is said to be unital.*
denote the unity by $\mathbf{1}$.

Further we consider only the case $m = 2$ and for the simplicity we use

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

for MSC, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ stand for any elements of $\mathbb{F}$.

Due to [1] we have the following classification theorems according to $Char(\mathbb{F}) \neq 2,3$, $Char(\mathbb{F}) = 2$ and $Char(\mathbb{F}) = 3$ cases, respectively.

Theorem 2.4 *Over an algebraically closed field* $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2$ *and* 3), *any non-trivial* 2*-dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constants:*

- $A_1(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 + 1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1 + 1 & -\alpha_2 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$,
- $A_2(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}; \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3$,
- $A_3(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$, *where* $\mathbf{c} = (\beta_1, \beta_2) \in \mathbb{F}^2$,
- $A_4(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{F}^2$,
- $A_5(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1-\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{F}$,
- $A_6(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}; \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{F}^2$,
- $A_7(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathbf{c} = \beta_1 \in \mathbb{F}$,
- $A_8(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathbb{F}$,

$$A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix},\ A_{10} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ A_{11} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.5 *Over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F})=2)$, *any non-trivial* 2-*dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constants:*

- $A_{1,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4\\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{F}^4$,
- $A_{2,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{F}^3$,
- $A_{3,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ 0 & \beta_2 & 1-\alpha_1 & 1\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{F}^2$,
- $A_{4,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{F}^2$,
- $A_{5,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 1 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,
- $A_{6,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{F}^2$,
- $A_{7,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & -1\end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,
- $A_{8,2}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,
- $A_{9,2}=\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\end{pmatrix}$, $A_{10,2}=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$, $A_{11,2}=\begin{pmatrix}1 & 1 & 1 & 0\\ 0 & -1 & -1 & -1\end{pmatrix}$, $A_{12,2}=\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$.

Theorem 2.6 *Over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F})=3)$, *any non-trivial* 2-*dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constant matrices:*

- $A_{1,3}(\mathbf{c})=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4\\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{F}^4$,
- $A_{2,3}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix};\ \begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ -\beta_1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{F}^3$,
- $A_{3,3}(\mathbf{c})=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1 & -1\end{pmatrix}$, *where* $\mathbf{c}=(\beta_1,\beta_2)\in\mathbb{F}^2$,
- $A_{4,3}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{F}^2$,
- $A_{5,3}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & -1-\alpha_1 & 1-\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=\alpha_1\in\mathbb{F}$,
- $A_{6,3}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix};\ \begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, *where* $\mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{F}^2$,

- $A_{7,3}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathbf{c} = \beta_1 \in \mathrm{F}$,
- $A_{8,3}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, *where* $\mathbf{c} = \alpha_1 \in \mathrm{F}$,
- $A_{9,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, $A_{10,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $A_{11,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$, $A_{12,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Remark 2.7 *In reality in [1] the class* $A_{3,2}(\mathbf{c})$ *should be understood as it is in this paper, as far as there is a type-mistake in this case in [1].*

3. Two-dimensional left unital algebras

Let $\mathbb{A}$ be a left unital algebra. In terms of its MSC A the algebra $\mathbb{A}$ to be left unital is written as follows:

$$A(l \otimes u) = u, \quad (2)$$

where $u = (u_1, u_2, ..., u_m)^T$, and $l = (t_1, t_2, ..., t_m)^T$ are column coordinate vectors of **u** and **1 L**, respectively.

It is easy to see that for a given 2-dimensional algebra $\mathbb{A}$ with MSC $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ the existence of a left unit element is equivalent to the equality of ranks of the matrices

$$M = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \\ \alpha_2 & \alpha_4 \\ \beta_2-\alpha_1 & \beta_4-\alpha_3 \end{pmatrix} \textit{ and } M' = \begin{pmatrix} \alpha_1 & \alpha_3 & 1 \\ \beta_1 & \beta_3 & 0 \\ \alpha_2 & \alpha_4 & 0 \\ \beta_2-\alpha_1 & \beta_4-\alpha_3 & 0 \end{pmatrix}.$$

This equality holds if and only if

$$\begin{vmatrix} \beta_1 & \beta_3 \\ \alpha_2 & \alpha_4 \end{vmatrix} = \begin{vmatrix} \beta_1 & \beta_3 \\ \beta_2-\alpha_1 & \beta_4-\alpha_3 \end{vmatrix} = \begin{vmatrix} \alpha_2 & \alpha_4 \\ \beta_2-\alpha_1 & \beta_4-\alpha_3 \end{vmatrix} = 0, \qquad (3)$$

and at least one of the following two cases holds true:

$$(\alpha_1, \alpha_3) \neq 0, (\beta_1, \beta_3) = (\alpha_2, \alpha_4) = (\beta_2-\alpha_1, \beta_4-\alpha_3) = 0, \qquad (4)$$

or

$$\begin{vmatrix} \alpha_1 & \alpha_3 \\ a & b \end{vmatrix} \neq 0, \textit{ whenever there exists nonzero } (a,b) \in \{(\beta_1, \beta_3), (\alpha_2, \alpha_4), (\beta_2-\alpha_1, \beta_4-\alpha_3)\}.$$

(5)

Note that the conditions (3), (4) and (3), (5) correspond to the existence of many and unique left units, respectively.

Theorem 3.1 *Over any algebraically closed field* F $(Char(\mathrm{F}) \neq 2$ *and* $3)$ *any nontrivial* 2*-dimensional left unital algebra is isomorphic to only one of the following non-isomorphic left unital algebras presented by their MSC:*

- $$A_1\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1} - \frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2} - \frac{1-\alpha_1}{2\beta_1}, \beta_1\right) = \begin{pmatrix} \alpha_1 & \frac{2\alpha_1 - 2\alpha_1^2 - \beta_1}{2\beta_1} & \frac{2\alpha_1 - 2\alpha_1^2 + \beta_1}{2\beta_1} & \frac{2\alpha_1 - 4\alpha_1^2 + 2\alpha_1^3 - \beta_1 + \alpha_1\beta_1}{2\beta_1^2} \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & \frac{-2\alpha_1 + 2\alpha_1^2 + \beta_1}{2\beta_1} \end{pmatrix},$$ where $\beta_1 \neq 0$,
- $$A_1\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right) = \begin{pmatrix} 1 & \alpha_2 & 1+\alpha_2 & \frac{1}{2}\left(\alpha_2 + 2\alpha_2^2\right) \\ 0 & -1 & 0 & -\alpha_2 \end{pmatrix},$$
- $$A_2(\alpha_1, 0, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$$ where $\alpha_1 \neq 0$,
- $$A_4(\alpha_1, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$$ where $\alpha_1 \neq 0$,
- $$A_6\left(\frac{1}{2}, 0\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}, \quad A_8\left(\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

We present the corresponding results in characteristic 2 and 3 cases without proof as follows.

Theorem 3.2 *Over any algebraically closed field* $\mathbb{F}$ *of characteristic* 2 *any nontrivial* 2*-dimensional left unital algebra is isomorphic to only one of the following non-isomorphic left unital algebras presented by their MSC:*

- $$A_{1,2}(\alpha_1, 0, \alpha_4, 0) = \begin{pmatrix} \alpha_1 & 0 & 1 & \alpha_4 \\ 0 & -\alpha_1 & 1-\alpha_1 & 0 \end{pmatrix},$$ where $\alpha_1 \neq 0$,
- $$A_{2,2}(\alpha_1, 0, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$$ where $\alpha_1 \neq 0$,
- $$A_{3,2}(1, \beta_2) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & \beta_2 & 0 & 1 \end{pmatrix},$$
- $$A_{4,2}(\alpha_1, \alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$$ where $\alpha_1 \neq 0$,
- $$A_{7,2}(0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad A_{10,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3.3 *Over any algebraically closed field* $\mathbb{F}$ *of characteristic* 3 *any nontrivial* 2*-dimensional left unital algebra is isomorphic to only one of the following non-isomorphic left unital algebras presented by their MSC:*

$$A_{1,3}\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1}-\frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2}-\frac{1-\alpha_1}{2\beta_1}, \beta_1\right)=$$

- $=\begin{pmatrix} \alpha_1 & \frac{2\alpha_1-2\alpha_1^2-\beta_1}{2\beta_1} & \frac{2\alpha_1-2\alpha_1^2+\beta_1}{2\beta_1} & \frac{2\alpha_1-4\alpha_1^2+2\alpha_1^3-\beta_1+\alpha_1\beta_1}{2\beta_1^2} \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & \frac{-2\alpha_1+2\alpha_1^2+\beta_1}{2\beta_1} \end{pmatrix},$ where $\beta_1 \neq 0$,

- $A_{1,3}\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right)=\begin{pmatrix} 1 & \alpha_2 & 1+\alpha_2 & \frac{1}{2}\left(\alpha_2+2\alpha_2^2\right) \\ 0 & -1 & 0 & -\alpha_2 \end{pmatrix},$

- $A_{2,3}(\alpha_1, 0, \alpha_1)=\begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$ where $\alpha_1 \neq 0$,

- $A_{4,3}(\alpha_1, \alpha_1)=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$ where $\alpha_1 \neq 0$,

- $A_{6,3}\left(\frac{1}{2}, 0\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},\ A_{8,3}\left(\frac{1}{2}\right)=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$

Note that according to Theorem 3.1 and Theorem 3.3 in the cases $Char(\mathrm{F}) \neq 2, 3$ and $Char(\mathrm{F}) = 3$ the lists are identical. Therefore, we summarize the final result for 2-dimensional left unital algebras in the following table, where all left units as well are given.

	Algebra	**1 L**
7*90 $Char(\mathrm{F}) \neq 2$	$A_1\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1}-\frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2}-\frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$, where $\beta_1 \neq 0$	$\frac{-2(1-\alpha_1)}{\beta_1}e_1+2e_2$
	$A_1\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right)$	$-(1+2\alpha_2)e_1+2e_2$
	$A_2(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\frac{1}{\alpha_1}e_1$
	$A_4(1,1)$	e_1+te_2, where $t \in \mathrm{F}$
	$A_4(\alpha_1, \alpha_1)$, where $\alpha_1 \neq 0,1$	$\frac{1}{\alpha_1}e_1$
	$A_6\left(\frac{1}{2}, 0\right)$	$2e_1$
	$A_8(\frac{1}{2})$	$2e_1$
6*90 $Char(\mathrm{F}) = 2$	$A_{1,2}(\alpha_1, 0, \alpha_4, 0)$, where $\alpha_1 \neq 0$	$\frac{1}{\alpha_1}e_1$

	$A_{2,2}(\alpha_1,0,\alpha_1)$, where $\alpha_1\neq 0$	$\frac{1}{\alpha_1}e_1$
	$A_{3,2}(1,\beta_2)$	e_2
	$A_{4,2}(1,1)$	e_1+te_2, where $t\in \mathrm{F}$
	$A_{4,2}(\alpha_1,\alpha_1)$, where $\alpha_1\neq 0,1$	$\frac{1}{\alpha_1}e_1$
	$A_{7,2}(0)$	e_2
	$A_{10,2}$	e_2

4. Two-dimensional right unital algebras

Now let us consider the existence of a right unit for an algebra $\mathbb{A}$ given by its MSC $A=$

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{pmatrix}.$$

It is easy to see that $\mathbb{A}$ has a right unit element if and only if the following matrices

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\\ \alpha_3 & \alpha_4\\ \beta_3-\alpha_1 & \beta_4-\alpha_2\end{pmatrix},\begin{pmatrix}\alpha_1 & \alpha_2 & 1\\ \beta_1 & \beta_2 & 0\\ \alpha_3 & \alpha_4 & 0\\ \beta_3-\alpha_1 & \beta_4-\alpha_2 & 0\end{pmatrix}$$

have equal ranks. It happens if and only if

$$\begin{vmatrix}\beta_1 & \beta_2\\ \alpha_3 & \alpha_4\end{vmatrix}=\begin{vmatrix}\beta_1 & \beta_2\\ \beta_3-\alpha_1 & \beta_4-\alpha_2\end{vmatrix}=\begin{vmatrix}\alpha_3 & \alpha_4\\ \beta_3-\alpha_1 & \beta_4-\alpha_2\end{vmatrix}=0$$

and at least one of the following two cases holds true

$$(\alpha_1,\alpha_2)\neq 0,(\beta_1,\beta_2)=(\alpha_3,\alpha_4)=(\beta_3-\alpha_1,\beta_4-\alpha_2)=0,$$

or

$$\begin{vmatrix}\alpha_1 & \alpha_2\\ a & b\end{vmatrix}\neq 0,\ \textit{if there exists nonzero}\ (a,b)\in\{(\beta_1,\beta_2),(\alpha_3,\alpha_4),(\beta_3-\alpha_1,\beta_4-\alpha_2)\}.$$

Because of similarity of proofs in right unital cases to those of left unital ones we present the result without proof by the following theorems.

Theorem 4.1 *Over any algebraically closed field* F *of characteristic not* 2 *any nontrivial* 2*-dimensional right unital algebra is isomorphic to only one of the following non-isomorphic right unital algebras presented by their MSC:*

- $A_1\left(\alpha_1,\frac{\alpha_1(1-2\alpha_1)}{2\beta_1},-\frac{\alpha_1^2(1-2\alpha_1)}{2\beta_1^2}-\frac{\alpha_1}{\beta_1},\beta_1\right)$, where $\alpha_1\beta_1\neq 0$,
- $A_1(0,\alpha_2,-2\alpha_2(1+\alpha_2),0)$, where $\alpha_2(1+\alpha_2)\neq 0$,
- $A_1\left(\frac{1}{2},-1,\alpha_4,0\right)$, $A_2\left(\frac{1}{2},0,\beta_2\right)$, $A_4\left(\frac{1}{2},\beta_2\right)$.

Theorem 4.2 *Over any algebraically closed field* $\mathbb{F}$ *of characteristic* 2 *any nontrivial* 2 *-dimensional right unital algebra is isomorphic to only one of the following non-isomorphic right unital algebras presented by their MSC:*

- $A_{1,2}(0,\alpha_2,0,\beta_1)$, where $\alpha_2 \neq 0$,
- $A_{3,2}(\alpha_1,0)$,
- $A_{6,2}(\alpha_1,0)$, where $\alpha_1 \neq 0$,
- $A_{7,2}(1)$,
- $A_{8,2}(\alpha_1)$, where $\alpha_1 \neq 0$,
- $A_{10,2}$.

The results obtained are summarized in the following table, where all right units as well are listed.

	Algebra	**1 R**
6*90 $Char(\mathbb{F}) \neq 2$	$A_1\left(\alpha_1, \frac{\alpha_1(1-2\alpha_1)}{2\beta_1}, \frac{-\alpha_1^2(1-2\alpha_1)}{2\beta_1^2} - \frac{\alpha_1}{\beta_1}, \beta_1\right)$, where $\alpha_1\beta_1 \neq 0$	$2e_1 + \frac{2\beta_1}{\alpha_1}e_2$
	$A_1(0,\alpha_2,-2\alpha_2(1+\alpha_2),0)$, where $\alpha_2(1+\alpha_2) \neq 0$	$2e_1 + \frac{1}{\alpha_2}e_2$
	$A_1\left(\frac{1}{2},-1,\alpha_4,0\right)$	$2e_1$
	$A_2(\frac{1}{2},0,\beta_2)$	$2e_1$
	$A_4(\frac{1}{2},0)$	$2e_1 + te_2$, where $t \in \mathbb{F}$
	$A_4(\frac{1}{2},\beta_2)$, where $\beta_2 \neq 0$	$2e_1$
7*90 $Char(\mathbb{F}) = 2$	$A_{1,2}(0,\alpha_2,0,\beta_1)$, where $\alpha_2 \neq 0$	$\frac{1}{\alpha_2}e_2$
	$A_{3,2}(\alpha_1,0)$	e_2
	$A_{6,2}(\alpha_1,0)$, where $\alpha_1 \neq 0$	$\frac{1}{\alpha_1}e_1$
	$A_{7,2}(1)$	e_2
	$A_{8,2}(1)$	$e_1 + te_2$, where $t \in \mathbb{F}$
	$A_{8,2}(\alpha_1)$, where $\alpha_1 \neq 0,1$	e_1
	$A_{10,2}$	e_2

Corollary 4.3 *Over an algebraically closed field* $\mathbb{F}$, $(Char(\mathbb{F}) \neq 2)$, *there exist, up to isomorphism, only two non-trivial* 2*-dimensional unital algebras given by their matrices of structure constants as follows*

$$A_2\left(\frac{1}{2},0,\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & 1\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix},\quad A_4\left(\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}.$$

Corollary 4.4 *Over an algebraically closed field* $\mathbb{F}$, $(Char(\mathbb{F}) = 2)$, *there exists, up to isomorphism, only one non-trivial* 2*-dimensional unital algebra given by its matrix of structure constants as*

$$A_{10,2}=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

5. Two-dimensional left and right unital real algebras

Due to [4] we have the following classification theorem.

Theorem 5.1 *Any non-trivial 2-dimensional real algebra is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:*

$$A_{1,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4\\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2\end{pmatrix},\ \textit{where}\ \mathbf{c}=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{R}^4,$$

$$A_{2,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix},\textit{where}\ \beta_1\geq 0,\ \mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{R}^3,$$

$$A_{3,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & -1\\ \beta_1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix},\textit{where}\ \beta_1\geq 0,\ \mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{R}^3,$$

$$A_{4,r}(\mathbf{c})=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1 & -1\end{pmatrix},\ \textit{where}\ \mathbf{c}=(\beta_1,\beta_2)\in\mathbb{R}^2,$$

$$A_{5,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix},\ \textit{where}\ \mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{R}^2,$$

$$A_{6,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix},\ \textit{where}\ \mathbf{c}=\alpha_1\in\mathbb{R},$$

$$A_{7,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},\textit{where}\ \beta_1\geq 0,\ \mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{R}^2,$$

$$A_{8,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & -1\\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},\textit{where}\ \beta_1\geq 0,\ \mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{R}^2,$$

$$A_{9,r}(\mathbf{c})=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 0 & -1\end{pmatrix},\ \textit{where}\ \mathbf{c}=\beta_1\in\mathbb{R},$$

$$A_{10,r}(\mathbf{c})=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},\ \textit{where}\ \mathbf{c}=\alpha_1\in\mathbb{R},$$

$$A_{11,r}=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 1 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix},\ A_{12,r}=\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & -1\end{pmatrix},$$

$$A_{13,r}=\begin{pmatrix}0 & 1 & 1 & 0\\ -1 & 0 & 0 & -1\end{pmatrix},\ A_{14,r}=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},\ A_{15,r}=\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}.$$

Owing to Theorem 5.1 the following results can be proved.

Theorem 5.2 *Over the real field* $\mathbb{R}$ *up to isomorphism there exist only the following nontrivial non-isomorphic two dimensional left unital algebras*

- $$A_{1,r}\left(\alpha_1,\frac{\alpha_1(1-\alpha_1)}{\beta_1}-\frac{1}{2},\frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2}-\frac{1-\alpha_1}{2\beta_1},\beta_1\right)=\begin{pmatrix}\alpha_1 & \frac{2\alpha_1-2\alpha_1^2-\beta_1}{2\beta_1} & \frac{2\alpha_1-2\alpha_1^2+\beta_1}{2\beta_1} & \frac{2\alpha_1-4\alpha_1^2+2\alpha_1^3-\beta_1+\alpha_1\beta_1}{2\beta_1^2}\\ \beta_1 & -\alpha_1 & 1-\alpha_1 & \frac{-2\alpha_1+2\alpha_1^2+\beta_1}{2\beta_1}\end{pmatrix},$$

  where $\beta_1\neq 0,$

- $A_{1,r}\left(1,\alpha_2,\frac{\alpha_2(2\alpha_2+1)}{2},0\right)=\begin{pmatrix}1 & \alpha_2 & 1+\alpha_2 & \frac{1}{2}\left(\alpha_2+2\alpha_2^2\right)\\ 0 & -1 & 0 & -\alpha_2\end{pmatrix},$
- $A_{2,r}(\alpha_1,0,\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ 0 & \alpha_1 & -\alpha_1+1 & 0\end{pmatrix},$ where $\alpha_1\neq 0,$
- $A_{3,r}(\alpha_1,0,\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & -1\\ 0 & \alpha_1 & -\alpha_1+1 & 0\end{pmatrix},$ where $\alpha_1\neq 0,$
- $A_{5,r}(\alpha_1,\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \alpha_1 & -\alpha_1+1 & 0\end{pmatrix},$ where $\alpha_1\neq 0,$
- $A_{7,r}\left(\frac{1}{2},0\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & 1\\ 0 & \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix},\ A_{8,r}\left(\frac{1}{2},0\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & -1\\ 0 & \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix},A_{10,r}\left(\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}.$

Theorem 5.3 *Over the real field* $\mathbb{R}$ *up to isomorphism there exist only the following nontrivial non-isomorphic two dimensional right unital algebras:*

- $A_{1,r}\left(\alpha_1,\frac{\alpha_1(1-2\alpha_1)}{2\beta_1},-\frac{\alpha_1^2(1-2\alpha_1)}{2\beta_1^2}-\frac{\alpha_1}{\beta_1},\beta_1\right),$ where $\alpha_1\beta_1\neq 0,$
- $A_{1,r}(0,\alpha_2,-2\alpha_2(1+\alpha_2),0),$ where $\alpha_2(1+\alpha_2)\neq 0,$
- $A_{1,r}\left(\frac{1}{2},-1,\alpha_4,0\right),\ A_{2,r}\left(\frac{1}{2},0,\beta_2\right),\ A_{3,r}\left(\frac{1}{2},0,\beta_2\right),A_{5,r}\left(\frac{1}{2},\beta_2\right).$

The results above are represented in the following table, where the units as well are shown.]

Algebra	**1 L**
$A_{1,r}\left(\alpha_1, \frac{\alpha_1(1-\alpha_1)}{\beta_1} - \frac{1}{2}, \frac{\alpha_1(1-\alpha_1)^2}{\beta_1^2} - \frac{1-\alpha_1}{2\beta_1}, \beta_1\right)$, where $\beta_1 \neq 0$	$\frac{-2(1-\alpha_1)}{\beta_1}e_1 + 2e_2$
$A_{1,r}\left(1, \alpha_2, \frac{\alpha_2(2\alpha_2+1)}{2}, 0\right)$	$-(1+2\alpha_2)e_1 + 2e_2$
$A_{2,r}(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\frac{1}{\alpha_1}e_1$
$A_{3,r}(\alpha_1, 0, \alpha_1)$, where $\alpha_1 \neq 0$	$\frac{1}{\alpha_1}e_1$
$A_{5,r}(1,1)$	$e_1 + te_2$, where $t \in \mathrm{R}$
$A_{5,r}(\alpha_1, \alpha_1)$, where $\alpha_1 \neq 0,1$.	$\frac{1}{\alpha_1}e_1$
$A_{7,r}\left(\frac{1}{2}, 0\right)$	$2e_1$
$A_{8,r}\left(\frac{1}{2}, 0\right)$	$2e_1$
$A_{10,r}(\frac{1}{2})$	$2e_1$
Algebra	**1 R**
$A_{1,r}\left(\alpha_1, \frac{\alpha_1(1-2\alpha_1)}{2\beta_1}, \frac{-\alpha_1^2(1-2\alpha_1)}{2\beta_1^2} - \frac{\alpha_1}{\beta_1}, \beta_1\right)$, where $\alpha_1\beta_1 \neq 0$	$2e_1 + \frac{2\beta_1}{\alpha_1}e_2$
$A_{1,r}(0, \alpha_2, -2\alpha_2(\alpha_2+1), 0)$, where $\alpha_2(1+\alpha_2) \neq 0$	$2e_1 + \frac{1}{\alpha_2}e_2$
$A_{1,r}\left(\frac{1}{2}, -1, \alpha_4, 0\right)$	$2e_1$
$A_{2,r}(\frac{1}{2}, 0, \beta_2)$	$2e_1$
$A_{3,r}(\frac{1}{2}, 0, \beta_2)$	$2e_1$
$A_{5,r}(\frac{1}{2}, 0)$	$2e_1 + te_2$, where $t \in \mathrm{R}$
$A_{5,r}(\frac{1}{2}, \beta_2)$, where $\beta_2 \neq 0$	$2e_1$

Corollary 5.4 *Up to isomorphism there are only the following nontrivial* 2 *-dimensional real unital algebras.*

$$A_{2,r}\left(\frac{1}{2},0,\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & 1\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix},\ A_{3,r}\left(\frac{1}{2},0,\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & -1\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix},$$

$$A_{5,r}\left(\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}.$$

Among these algebras only $A_{3,r}\left(\frac{1}{2},0,\frac{1}{2}\right)$ is a division algebra and it is isomorphic to the algebra of complex numbers.

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