ON STABILITY FOR ONE PARABOLIC TYPE PROBLEM

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Abstract. *The article considers one boundary value problem for parabolic type systems, consisting of two differential equations for determining the temperature of raw cotton components during drying in drying units. An approximate solution of the Galerkin method to the problem under consideration is constructed. The stability of the Galerkin method for an approximate solution of the problem under consideration has been established under the condition of a strongly minimal coordinate system.*

Keywords: *mathematical model, algorithm, temperature, coordinate system, monotonicity, stability, strongly minimal.*

INTRODUCTION: A lump is raw cotton that has a small volume and a ball shape. During the drying process, a complex non-stationary heat and mass transfer process occurs in a lump of wet raw cotton, which determines the external and internal states. External processes are characterized by mass transfer from the surface of the lump into the environment and heat exchange between the fiber and the environment. The rate of heat propagation in a bolus of raw cotton is important for maintaining the quality of fiber and seeds during drying [1-5].

The problem of heat and mass transfer in a lump of raw cotton can be expressed in the form of a system of differential equations of parabolic type:

$$\begin{cases} \dfrac{\partial u_1}{\partial \tau} = a_1\left(\dfrac{\partial^2 u_1}{\partial r^2} + \dfrac{2}{r}\cdot\dfrac{\partial u_1}{\partial r}\right) + \dfrac{\alpha}{c_1\rho_1}[u_2 - u_1] + \dfrac{1}{c_1\rho_1} f_1(\tau) \\ \dfrac{\partial u_2}{\partial \tau} = a_2\left(\dfrac{\partial^2 u_2}{\partial r^2} + \dfrac{2}{r}\cdot\dfrac{\partial u_2}{\partial r}\right) + \dfrac{\alpha}{c_2\rho_2}[u_1 - u_2] + \dfrac{1}{c_2\rho_2} f_2(\tau) \end{cases} \tag{1}$$

From the beginning

$$u_1(r, 0)=g_1(r),\ u_2(r,0)=g_2(r), \tag{2}$$

and boundary conditions: at $r=R$

$$\begin{aligned} -\frac{\partial u_1}{\partial r} + \alpha_1[u_B - u_1] = 0 \\ -\frac{\partial u_2}{\partial r} + \alpha_2[u_B - u_2] = 0 \end{aligned} \tag{3}$$

where α_1- is the heat exchange coefficient between the seed and the coolant; α_2- is the heat transfer coefficient between the fiber and the coolant; α- is the heat exchange coefficient between the seed and the fiber; $f_1(\tau)$, $f_2(\tau)$- specified continuous functions describing the process of moisture evaporation from the seed and from the fiber, respectively, u_B- air temperature, τ- time of the drying process; $u_1(r, \tau)$, $u_2(r, \tau)$- are the required functions representing the temperature of the seed and fiber at point r at a given time τ.

By making a change of variables

$$\theta_i(r, \tau) = r \cdot u_i(r, \tau), \qquad i = 1,2$$

we obtain a system of equations in the form:

$$\begin{cases} \dfrac{\partial\theta_1}{\partial\tau} = a_1\dfrac{\partial\theta_1^2}{\partial r^2} + \alpha(\theta_2 - \theta_1) + \dfrac{1}{c_1\rho_1} r f_1(\tau) \\ \dfrac{\partial\theta_2}{\partial\tau} = a_2\dfrac{\partial\theta_2}{\partial r^2} + \alpha(\theta_1 - \theta_2) + \dfrac{1}{c_2\rho_2} r f_2(\tau) \end{cases} \tag{4}$$

with initial conditions

$$\theta_1(r,0) = r \cdot g_1(r), \qquad \theta_2(r,0) = r \cdot g_2(r) \tag{5}$$

and boundary conditions at $r=R$

$$\begin{aligned} -\frac{\partial\theta_1(R,\tau)}{\partial r} + \frac{1}{R}\cdot\theta_1(R,\tau) + \alpha_1(u_B - \theta_2(R,\tau)) = 0 \\ -\frac{\partial\theta_2(R,\tau)}{\partial r} + \frac{1}{R}\cdot\theta_2(R,\tau) + \alpha_2(u_B - \theta_2(R,\tau)) = 0 \end{aligned} \tag{6}$$

where $\tau \in [0; T]$, $\quad r \in [0; R]$.

To solve this problem, we will use the Bubnov-Galerkin projection method, i.e. Let us introduce a basis function $\varphi_i(x)$ satisfying the conjugacy condition $\varphi_i(0)=0$, $\varphi_i(r) \in W_2^1(0,R)$. Then we will look for a solution to system (4) – (6) in the form [6-10].

$$\theta_{iN}(r,\tau) = \sum_{K=1}^{N} a_{iK}(\tau)\cdot\varphi_K(r) \qquad (i = 1,2) \tag{7}$$

Where coefficicents $a_i(\tau)$ $(i=1,2)$ are determined from

$$\begin{cases} \left(\dfrac{\partial\theta_{1N}}{\partial\tau} - a_1\dfrac{\partial^2\theta_{1N}}{\partial r^2} - \dfrac{\alpha}{c_1\rho_1}(\theta_{2N} - \theta_{1N}) - \dfrac{1}{c_1\rho_1} r\cdot f_1(\tau), \varphi(r)\right)_2 = 0 \\ \left(\dfrac{\partial\theta_{2N}}{\partial\tau} - a_2\dfrac{\partial^2\theta_{2N}}{\partial r^2} - \dfrac{\alpha}{c_2\rho_2}(\theta_{1N} - \theta_{2N}) - \dfrac{1}{c_2\rho_2} r\cdot f_2(\tau), \varphi(r)\right)_2 = 0 \end{cases} \tag{8}$$

Denoting by

$$\alpha_{ik} = (\varphi_i(r), \varphi_j(r))_2;$$

$$\beta_{ik} = a_1(\varphi_i'(r), \varphi_k'(r))_2 + a_1\left(\frac{I}{R} - H_1\right)\varphi_i(R)\cdot\varphi_k(R) + H_3\cdot\alpha_{ik};$$

$$\overline{\beta}_{ik} = a_2(\varphi_i'(r), \varphi_k'(r))_2 + a_2\left(\frac{I}{R} - H_2\right)\varphi_i(R)\cdot\varphi_k(R) + H_4\cdot\alpha_{ik};$$

$$\gamma_{ik} = H_3\cdot d_{ik}; \qquad \overline{\gamma}_{ik} = H_4\cdot d_{ik}$$

$$f_{1i} = (f_1(\tau)\cdot r, \varphi_i(r))_2 + a_1 H_1 R\, u_B\cdot\varphi_i(R)$$

$$f_{2i} = (f_2(\tau)\cdot r, \varphi_i(r))_2 + a_2 H_2 R\, u_B\cdot\varphi_i(R)$$

we obtain the following differential equations:

$$\begin{cases} Q_n \cdot \dfrac{dA_n(\tau)}{d\tau} + P_n A_{1n}(\tau) + G_n A_{2n}(\tau) = F_{1n}(\tau, r) \\ Q_n \cdot \dfrac{dB_n(\tau)}{d\tau} + \widetilde{P}_n A_{2n}(\tau) + \widetilde{G}_n A_{1n}(\tau) = F_{2n}(\tau, r) \\ Q_n A_n(0) = F_{10}(r) \\ Q_n B_n(0) = F_{20}(r) \end{cases} \tag{9}$$

where $Q_n=(\alpha_{ik})$, $P_n=(\beta_{ik})$, $G_n=(\gamma_{ik})$, $\widetilde{P}_n = (\widetilde{\beta}_{ik})$ и $\widetilde{G}_n = (\widetilde{\gamma}_{ik})$ square matrices of size (NxN);

$A_{1n}(\tau)=(a_{11}(\tau), a_{12}(\tau), \ldots, a_{1n}(\tau))^T$, $A_{2n}(\tau)=(a_{21}(\tau), a_{22}(\tau), \ldots, a_{2n}(\tau))^T$ – required vectors;

$F_{1n}(\tau,r)=(f_{11}(\tau,r), f_{12}(\tau,r), \ldots, f_{1n}(\tau,r))^T$, $F_{2n}(\tau,r)=(f_{21}(\tau,r), f_{22}(\tau,r), \ldots, f_{2n}(\tau,r))^T$ known vectors, elements that are determined by the above formulas;

vector elements

$F_{10}(r)=(f_{01}(r), f_{02}(r), \ldots, f_{0n}(r))^T$ and $F_{20}(r)=(\widetilde{f}_{01}(r), \widetilde{f}_{02}(r), \ldots, \widetilde{f}_{0n}(r))^T$ are determined from the relation $f_{0i} = (rg_1(r), \varphi_i(r))$, $\widetilde{f}_{0i} = (rg_2(r), \varphi_i(r))$.

Let us investigate the question of the stability of problem (9). Let us assume that the coordinate system $\{\varphi_i(x)\}$ is strongly minimal in the space $L_2(\Omega)$ i.e. there is a constant independent of n, such that , $0 < q < q_i^n$, where q_i^n - are the eigenvalues of the matrix [11-13].

$$Q_n = \{(\varphi_k, \varphi_j)\}_{k,\, j=1}^{n}$$

Let us assume that instead of the Galerkin system (9) we solve the "perturbed" system

$$\begin{cases} (Q_n + \Gamma_n)\widetilde{C}_n(\tau) + (P_n + \Gamma'_n)\widetilde{C}_n = F_n(\tau) + \varepsilon_n \\ (Q_n + \Gamma_n^0)\widetilde{C}_n(0) = F_0 + \varepsilon_0 \end{cases} \tag{10}$$

where $\widetilde{C}_n(\tau) = (\widetilde{A}_{1n}(\tau); \widetilde{A}_{2n}(\tau))$ - solution to the perturbed problem.

The Galerkin process for problem (4) - (6) is called stable if there are such independent n positive constants p_i, that for sufficiently small matrix norms $\|\Gamma_n^0\|, \|\Gamma_n\|, \|\Gamma'_n\|$ and norms of vectors $\|\varepsilon_0\|, \|\varepsilon_n\|$

$$\|\widetilde{C}_n(\tau) - C_n(\tau)\|_{E_n} \le p_0\|\varepsilon_0\| + p_1\|\varepsilon_n\| + p_2\|\Gamma_n^0\| + p_3\|\Gamma_n\| + p_4\|\Gamma'_n\| \tag{11}$$

The approximate solution $U(r,\tau)=(\theta_{1n}, \theta_{2n}(r,\tau))^T$ of system (4)-(6) is called stable in the space $L_2(\Omega)$, if an inequality similar to (11) holds for differences $\|\widetilde{U}(r,\tau) - U(r,\tau)\|$,

where $\widetilde{U}(r,\tau) = (\widetilde{\theta}_{1n}(r,\tau), \widetilde{\theta}_{2n}(r,\tau))^T$, $\quad \widetilde{\theta}_{in}(r,\tau) = \sum_{k=1}^{n} \widetilde{a}_{iK}(\tau) \cdot \varphi_K(r)$

By multiplying each of the equations of system (8) by the corresponding a1i $a_{1i}(\tau)$, $a_{2i}(\tau)$ and making similar calculations, we can establish the continuous dependence of the approximate solutions on the initial data and right-hand sides in the form [14-17]:

$$\begin{cases} \|\theta_{1n}\|_2^2 + \int_0^\tau \|\nabla\theta_{1n}\|_2^2 \le p_1\left[u_B^2 + \|g_1(r)\|_2^2 + \|g_2(r)\|_2^2 + \int_0^\tau f_1^2(\tau)d\tau + \int_0^\tau f_2^2(\tau)d\tau\right] \\ \|\theta_{2n}\|_2^2 + \int_0^\tau \|\nabla\theta_{2n}\|_2^2 \le p_2\left[u_B^2 + \|g_2(r)\|_2^2 + \|g_1(r)\|_2^2 + \int_0^\tau f_1^2(\tau)d\tau + \int_0^\tau f_2^2(\tau)d\tau\right] \end{cases} \quad (12)$$

In addition, taking into account the continuity of the given functions and using the mean value theorem, we can estimate

$$\|\theta_{1n}\|^2 + \int_0^\tau \|\nabla\theta_{1n}\|^2 \le M_1, \qquad \|\theta_{2n}\|^2 + \int_0^\tau \|\nabla\theta_{2n}\|^2 \le M_2$$

From here, using the strong minimality of basis functions in $L_2(\Omega)$, we can obtain the following inequalities:

$$\|G_n(\tau)\|_{E_n}^2 \le \frac{1}{q}\|U(r,\tau)\|_{L_2(\Omega)}^2 \le M, \quad \int_0^\tau \|\dot{G}_n(\tau)\|_{E_n}^2 d\tau' \le \frac{1}{q}\int_0^\tau \left\|\frac{\partial U(r,\tau)}{\partial\tau}\right\|_{L_2(\Omega)}^2 d\tau' \le K, \quad (13)$$

where $G_n(\tau) = (a_{1i}(\tau),\ a_{2i}(\tau))^T$

Let the mistakes made $\Gamma_n,\ \Gamma'_n$

$$\|\Gamma_n\| \le e_1 q, \quad \|\Gamma'_n\| \le e_2 q; \quad 0 \le e_i \le 1, \quad q > 0. \quad (14)$$

Let's denote by $Z_n(\tau) = \tilde{G}_n(\tau) - G_n(\tau)$. From the system of equations (10) we subtract the system of equations (9). Let us multiply the resulting equation scalarly by $\dot{Z}_n(\tau)$, i.e.

$$\frac{1}{2}\frac{d}{d\tau}((Q_n + \Gamma_n)Z_n, Z_n) + ((P_n + \Gamma'_n)Z_n,\ \dot{Z}_n) = (\varepsilon_n,\ \dot{Z}_n) + (\Phi_n,\ \dot{Z}_n) \quad (15)$$

where $\Phi_n(\tau) = -\Gamma_n \cdot \dot{G}(\tau) - \Gamma'_n\ G_n(\tau)$.

As matrice P_n is positive definite, then

$$((P_n + \Gamma'_n)Z_n,\ \dot{Z}_n)_n \ge 0.$$

Then, evaluating the terms on the right side of the equality

$$\left|(\varepsilon_n,\ \dot{Z}_n)_{E_n}\right| \le \frac{1}{2\varepsilon_1}\|\varepsilon_n\|^2 + \frac{1}{2}\varepsilon_1\|Z_n\|^2$$

and
$$\left|(\Phi_n(\tau),\ \dot{Z}_n)\right| \le \frac{1}{2\varepsilon_1}\|\Phi_n(\tau)\|^2 + \frac{1}{2}\varepsilon_1\|Z_n\|^2$$

we give

$$\frac{1}{2}\frac{d}{d\tau}((Q_n + \Gamma_n)Z_n, Z_n) \le \varepsilon_1\|Z_n\|^2 + c_1\left(\|\varepsilon_n\|^2 + \|\Phi_n(\tau)\|^2\right)$$

Let us integrate the last inequalities over τ. Taking into account the inequality $\|Z_n\|_{E_n}^2 \le \frac{1}{2}\|\tilde{U} - U\|_{L_2}^2$

We give

$$\left(\left(Q_n+\Gamma_n\right)Z_n,\, Z_n\right)_{E_n} \le 2\varepsilon_1\int_0^\tau \left\|\widetilde{U}-U\right\|_{L_2}^2 d\tau + c_1\int_0^\tau \left(\left\|\varepsilon_n\right\|^2+\left\|\Phi_n(\tau)\right\|_{L_2}^2\right)d\tau + \left(\left(Q_n+\Gamma_n\right)Z_n(0),\, Z_n(0)\right)_{E_n} \quad (16)$$

On the other hand, due to assumption (14) we obtain

$$\left(\left(Q_n+\Gamma_n\right)Z_n,\, Z_n\right)_{E_n} \ge \left(Q_n Z_n,\, Z_n\right) - e_1 q\left\|Z_n\right\|_{E_n}^2 \ge \left(1-e_1\right)\left\|\widetilde{U}_n-U_n\right\|_{L_2}^2,$$

$$\left(\left(Q_n+\Gamma_n\right)Z_n(0),\, Z_n(0)\right)_{E_n} \le \left(Q_n Z_n(0),\, Z_n(0)\right)_{E_n} + e_1 q\left\|Z_n\right\|_{E_n}^2 \le c_2\left(1+e_1\right)\left\|\widetilde{U}(r,\,0)-U(r,\,0)\right\|_{L_2(\Omega)}^2$$

$$\int_0^\tau \left\|\Phi_n(\tau)\right\|_{E_n}^2 d\tau' \le 2M\int\left\|\Gamma_n\right\|^2 d\tau' + 2K\int_0^\tau \left\|\Gamma_n'\right\|^2 d\tau' \le 2MT\left\|\Gamma_n\right\|^2 + 2KT\left\|\Gamma_n'\right\|_{E_n}^2$$

Denote $\int_0^\tau \left\|\widetilde{U}_n(r,\,\tau)-\dot{U}_n(r,\,\tau)\right\|_{L_2}^2 d\tau = y(\tau)$

$$F_n(\tau)=c_2\left(1+e_1\right)\left\|\widetilde{U}_n(r,\,0)-U_n(r,\,0)\right\|_2^2 + c_1\left\|\varepsilon_n\right\|_{E_n}^2 + 2MT\left\|\Gamma_n\right\|_{E_n}^2 + 2KT\left\|\Gamma_n'\right\|_{E_n}^2$$

Then, substituting all estimates into (16), we obtain a differential inequality for $y_n(\tau)$.

$$\frac{dy_n(\tau)}{d\tau} \le M\cdot y_n(\tau)+F_n(\tau)$$

from which, in turn, by virtue of the theorem on differential inequalities follows the inequality

$$\frac{dy_n(\tau)}{d\tau} \le e^{G_1\tau}\cdot F(\tau)$$

Hence, using estimates (5) obtained for the initial conditions, we have:

$$\left\|\widetilde{U}_n(r,\,\tau)-U_n(x,\,\tau)\right\|_2^2 \le p_0\left\|\varepsilon_0\right\|^2 + p_1\left\|\varepsilon_n\right\|^2 + p_2\left\|\Gamma_n^0\right\|^2 + p_3\left\|\Gamma_n'\right\|^2 + p_4\left\|\Gamma_n'\right\|^2 \quad (17)$$

where $p_i\left(i=\overline{0,4}\right)$ do not depend on N. Hence,

$$\left\|\widetilde{G}_n(\tau)-G_n(\tau)\right\|_{E_n}^2 \le \frac{1}{q}\left\|\widetilde{U}_n(r,\,\tau)-U_n(r,\,\tau)\right\|_2^2 \le \frac{1}{q}\omega^2$$

where ω^2- is the right-hand side of inequality (17). The last relations imply the stability of the algorithm for constructing an approximate solution and the numerical stability of the approximate solution in $L_2(\Omega)$.

CONCLUSION. An approximate solution to the boundary value problem of parabolic type equations is constructed. The stability of the Galerkin method for an approximate solution of the problem under consideration has been established under the condition of a strongly minimal coordinate system.

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