SOLVING OF SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER BY MEANS OF THE METHOD OF ADDITIONAL ARGUMENT

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Abstract. *The application of the* method of additional argument *to a system of high-order partial integro-differential equations is relevant. Kyrgyz scientists have considered applications of this method to a system of partial differential equations of first order. In this paper, the system of partial integro-differential equations of the second order with initial conditions is reduced to a system of integral equations.*
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Investigations of various classes of systems of partial differential equations of the first order on the base of the method of additional argument were considered in [1-3]. Let $\overline{C}^{(k)}(G_2(T))$ be the class of functions being continuous and bounded together with their derivatives up to the k-th order in $G_2(T)=[0,T]\times R$.

Considered the system of partial differential equations of the second order of type

$$\frac{\partial^2 u_i(t,x)}{\partial t^2} = k_i^{\,2}(t,x)\frac{\partial^2 u_i(t,x)}{\partial x^2} + a_i(t,x)\frac{\partial u_i(t,x)}{\partial t} + b_i(t,x)\frac{\partial u_i(t,x)}{\partial x} + f_i(t,x,u_1,u_2), \qquad (1)$$
$$i=1,2, \quad (t,x)\in G_2(T),$$

with the initial conditions

$$\left.\frac{\partial^k u_i(t,x)}{\partial t^k}\right|_{t=0} = u_i^k(x), \quad i,k=0,1, \quad x\in R. \qquad (2)$$

where

$$f_i(t,x,u_1,u_2) = \psi_i(t,x,u_1,u_2) + \int_0^t K_i(t,s)u_i(s,x)ds,$$

$$u_i^k(x)\in \overline{C}^{\,(2-k)}(R),\ b_i(t,x), a_i(t,x)\in \overline{C}^{(2)}(G_2(T)),$$

$$\psi_i(t,x,u_1,u_2)\in \overline{C}^{(2)}(G_2(T)\times R^2),$$

$$K_i(t,s)\in C(G), \quad \int_0^T |K_i(t,s)|ds \le \gamma = const, \ (i,k=0,1).$$

The paper proposes a new way of reducing the system of partial integro-differential equations to the system of integral equations. Let us introduce some notation. Using method of additional argument.

In this paper, to reduce problem (1), (2) to a system of integral equations, the following notation is used:

Through $p_i(s,t,x)$, $q_i(s,t,x)$, $i=1,2$ denote, respectively, the solutions of the following integral equations:

$$p_i(s,t,x)=x+\int_s^t k_i(\nu,p_i(\nu,t,x))d\nu, \tag{3.}$$

$$q_i(s,t,x)=x-\int_s^t k_i(\nu,q_i(\nu,t,x))d\nu, \tag{4}$$

$i=1,2,\quad (s,t,x)\in Q_2(T)$,

$$Q_2(T)=\{(s,t,x)|\ 0\le s\le t\le T,\ x\in R\};$$

Differential operator:

$$D[\omega]=\frac{\partial}{\partial t}+\omega\frac{\partial}{\partial x};$$

New functions

$$\vartheta_i(t,x)=D[-k_i(t,x)]u_i(t,x),\quad i=1,2, \tag{5}$$

$$g_i(t,x)=\frac{1}{k_i(t,x)}[b_i(t,x)-\frac{\partial k_i(t,x)}{\partial t}-k_i(t,x)\frac{\partial k_i(t,x)}{\partial x}],\quad i=1,2,$$

$$\beta_i^1(t,x)=a_i(t,x)+g_i(t,x),\quad \beta_i^2(t,x)=a_i(t,x)-g_i(t,x),\quad \beta_i^3(t,x)=D[k_i(t,x)]\beta_i^1(t,x),\quad i=1,2.$$

Lemma 1. The system of partial integro-differential of second order (1) with the initial condition (2) is equivalent to the following system of integral equations:

$$\begin{aligned}\vartheta_i(t,x)&=\frac{1}{2}\varphi_i(q_i(0,t,x))+\frac{1}{2}\beta_i^1(t,x)u_i+\frac{1}{2}\int_0^t\beta_i^2(s,q_i)\vartheta_i(s,q_i)ds-\\&-\frac{1}{2}\int_0^t\beta_i^3(s,q_i)u_i(s,q_i)ds+\int_0^t f_i(s,q_i,u_1(s,q_i),u_2(s,q_i))ds,\end{aligned} \tag{6}$$

$$u_i(t,x)=u_i^0\left(p_i(0,t,x)\right)+\int_0^t\vartheta_i\left(s,p_i(s,t,x)\right)ds,\ \text{i=1,2}, \tag{7}$$

where

$$\left[2\vartheta_i(t,x)-\beta_i^1(t,x,u_i)u_i(t,x)\right]_{t=0}=\varphi_i(x),\quad i=1,2..$$

Proof. First, differentiating (6), (7), we prove that the system of integral equations (6), (7) satisfy equation (1) and initial condition (2).

From (6) we have:

$$\frac{\partial\vartheta_i(t,x)}{\partial t}+k_i(t,x)\frac{\partial\vartheta_i(t,x)}{\partial x}=a_i(t,x)\frac{\partial u_i(t,x)}{\partial t}+k_i(t,x)g_i(t,x)\frac{\partial u_i(t,x)}{\partial x}+f_i(t,x,u_1,u_2), \tag{8}$$

$i=1,2.$

Taking into account (5), from (8) we obtain (1). Therefore, (2) also holds.

Let us now show that, in turn, the solution to problem (1), (2) is a solution to system of integral equations (6)–(7). To do this, we write equation (1) in the form

$$D[k_i(t,x)]z_i(t,x;u_i)=\beta_i^2(t,x)\vartheta_i(t,x)-\beta_i^3(t,x)u_i(t,x)+2f_i(t,x,u_1,u_2),\qquad i=1,2, \tag{9}$$

where

$$z_i(t,x;u_i)=2\vartheta_i(t,x)-\beta_i^1(t,x)u_i(t,x),\quad i=1,2.$$

The solution of problem (9), (2) by the method of an additional argument is reduced to the integral equation (6). It follows from the notation (3) that (7)

In equation (6), substituting (7), we obtain an integral equation with respect to

$$\vartheta_i(t,x), \quad i=1,2.$$

$$\vartheta_i(t,x) = A_i(t,x;\vartheta_1,\vartheta_2) \equiv \frac{1}{2}\varphi_i(q_i(0,t,x)) + \frac{1}{2}\beta_i^1(t,x)\left(u_i^0(p_i(0,t,x)) + \int_0^t \vartheta_i(s,p_i(s,t,x))ds\right) +$$
$$+\frac{1}{2}\int_0^t \beta_i^2(s,q_i)\vartheta_i(s,q_i)ds - \frac{1}{2}\int_0^t \beta_i^3(s,q_i)\left(u_i^0(p_i(0,s,q_i)) + \int_0^s \vartheta_i(v,p_i(v,s,q_i))dv\right)ds +$$
$$+\int_0^t f_i(s,q_i,u_1^0(p_1(0,s,q_1)) + \int_0^s \vartheta_1(v,p_1(v,s,q_1))dv, u_2^0(p_2(0,s,q_2)) + \int_0^s \vartheta_2(v,p_2(v,s,q_2))dv))ds,$$

$i=1,2.$ (10)

Lemma 2. There is such a $T^* > 0,$ that the integral equation (10) has a unique solution in

$$\overline{C}(G_2(T^*)).$$

Proof. Let us show that Eq. (10) has a unique, continuous solution in the domain $G_2(T)$ at $T < T_*$ that satisfies the inequality

$$\|\vartheta_i - \phi_i\| \le M,$$

$$\phi_i \equiv \frac{1}{2}\varphi_i(q_i(0,t,x)) + \frac{1}{2}\beta_i^1(t,x)u_i^0(p_i(0,t,x)) - \frac{1}{2}\int_0^t \beta_i^3(s,q_i)u_i^0(p_i(0,s,q_i))ds,$$

$i=1,2.$ (10)

Let us show that, for $T < T_*$, the operators Ai are contraction operators

$$\|A_i\vartheta - \phi_i\| \le (M_0K + \|f\|)T + NK\frac{T^2}{2} = \Omega_0(T),$$

where

$$\|\vartheta_i\| \le \|\phi_i\| + M = K, \quad i=1,2.$$

$$\left|\beta_i^k(t,x)\right| \le M_0 = const, \;\; i,k=1,2, \quad \beta_i^3(t,x) \le N = const, i=1,2.$$

Denote by $T_0 -$ the positive root of the equation $\Omega_0(T) = M.$

It remains for us to show that the operators A_i compresses the distance between elements.

The following estimate is valid

$$\left\|A_i\vartheta^1 - A_{i1}\vartheta^2\right\| \le \Omega_1(T)\left\|\vartheta^1 - \vartheta^2\right\|,$$

where

$$\Omega_1(T) = M_0T + \left[\frac{N}{2} + L_1 + L_2 + \gamma\right]\frac{T^2}{2}$$

$$\left|\psi_i(t,x,u_1^1,u_2^1)-\psi_i(t,x,u_1^2,u_2^2)\right| \le L_1\left|u_1^1-u_1^2\right|+L_2\left|u_2^1-u_2^2\right| \quad L_i \ge 0, \quad L_i - const, \quad i=1,2.$$

Denote by T_1 the positive root of the equation $\Omega_1(T)=1$.

It follows from this that the operators A_i under $T<T^*=\min\{T_0\ ,T_1\}$ perform a contraction mapping. Then the equation defines a unique solution that belongs to this ball. This solution can be obtained by the method of successive approximations.

Conclusion. Using the proposed scheme of reduction to an integral equation, one can construct solutions of linear partial differential equations of the second order with given initial conditions.

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