INVESTIGATION OF SOLUTIONS OF PARTIAL INTEGRO-DIFFERENTIAL EQUATION OF FOURTH ORDER BY MEANS OF A NEW WAY

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Abstract. *This article discusses the partial integro-differential equation of fourth order. To solve the problem, a new method is used, with the help of which we will bring the partial integro-differential equation of fourth order to a form convenient for using the additional argument method. The initial problem for a fourth-order equation is first reduced to a system of second-order partial differential equations. And the system of equations in partial derivatives of the second order is reduced by the method of an additional argument to a system of integral equations, for which the principle of compressive reflections is applied.*
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Introduction.

Currently, this method of additional argument (MAA) is used to solve partial differential equations of various orders and systems of partial differential equations. Thus, MAA shows an advantage over the characteristic method in constructing a solution to high-order partial differential equations.

In [1,2] initial value problem for differential equation of the second order is considered. In [3] a new way to build solutions of partial differential equations of the fourth order of hyperbolic type is considered. We use results of this work.

Formulation of the problem.

We use denotations, classes and spaces of functions [1]: G2(T) := [0, T] × R; G := [0, T] × [0, T]; C(k) be the class of functions being continuous and bounded together with their derivatives up to the k-th order.

In this work we consider the following task

$$u_{tttt}(t,x)-2a^2u_{ttxx}(t,x)+a^4u_{xxxx}(t,x)=bu_{ttt}(t,x)+cu_{ttx}(t,x)+du_{txx}(t,x)+$$
$$+eu_{xxx}(t,x)+\int_0^t K(t,s)u(s,x)ds+f(t,x,u), \qquad (1)$$

with the initial conditions

$$\frac{\partial^k u(0,x)}{\partial t^k}=u_k(x), \quad k=0,1,2,3, \qquad (2)$$

where

$$a,b,c,d,e-const,\ f(t,x,u)\in \overline{C}^{(4)}(G_2(T)\times R^2),\ G_2(T)=[0,T]\times R.$$

$$K(t,s)\in C(G), \quad \int_0^T |K(t,s)|ds \le \gamma = const.$$

Let in the equation (1): $d = a^2 b,\ e = a^2 c.$

In equation (1), we introduce the notation:

$$\omega(t,x) = u_{tt}(t,x) - a^2 u_{xx}(t,x) - bu_t(t,x) - cu_x(t,x),$$

we reduce equation (1) to a system of equations of the form:

$$\begin{cases} \omega_{tt}(t,x) - a^2\omega_{xx}(t,x) = \int_0^t K(t,s)u(s,x)ds + f(t,x,u), \\ u_{tt}(t,x) - a^2 u_{xx}(t,x) = bu_t(t,x) + cu_x(t,x) + \omega(t,x). \end{cases} \tag{3}$$

From the first equation of system (3), using MAA, we express the function w in terms of u, then, substituting it into the second equation of the system, we find the required function u (t, x).

Consider the first equation of system (3):

$$\frac{\partial^2\omega(t.x)}{\partial t^2} = a^2\frac{\partial^2\omega(t,x)}{\partial x^2} + \int_0^t K(t,s)u(s,x)ds + f(t,x,u) \tag{4}$$

with the initial conditions (2).

Let us introduce the notation:

$$p(s,t,x) = x + at - as,$$
$$q(s,t,x) = x - at + as$$

We write the integro-differential equation (4) in the form:

$$\left(\frac{\partial\omega}{\partial t} - a\frac{\partial\omega}{\partial x}\right)_t + a\left(\frac{\partial\omega}{\partial t} - a\frac{\partial\omega}{\partial x}\right)_x = \int_0^t K(t,s)u(s,x)ds + f(t,x,u). \tag{5}$$

With the help of MAA from (5) we have:

$$\frac{\partial\omega}{\partial t} - a\frac{\partial\omega}{\partial x} = \omega_1(q(0,t,x)) + \int_0^t\int_0^\tau K(\tau,s)u(s,q(\tau,t,x))dsd\tau + $$
$$+\int_0^t f(\tau,q(\tau,t,x),u(\tau,q(\tau,t,x))d\tau, \tag{6}$$

Where $\omega_1(x) = (\omega_t - a\omega_x)|_{t=0}$, and is determined from the initial condition (2).

Now we apply MDA for (6) with (2).

Therefore, we get:

$$\omega(t,x) = \omega_0(p(0,t,x)) + \int_0^t \omega_1(q(0,s,p(s,t,x))ds +$$
$$+\int_0^t\int_0^\nu\int_0^\tau K(\tau,s)u(s,q(\tau,\nu,p(\nu,t,x)))dsd\tau d\nu + \int_0^t\int_0^\nu f(\tau,q(\tau,\nu,p),u(\tau,q(\tau,\nu,p))d\tau d\nu =$$
$$= F(t,x;u) \tag{7}$$

Now we solve the second equation of system (3) using the results of [1].

$$u_{tt}(t,x)=a^2u_{xx}(t,x)+bu_t(t,x)+cu_x(t,x)+F(t,x;u). \quad (8)$$

Let's use the notation:

$$\vartheta(t,x)=D[-a]u(t,x), \quad (9)$$

$$g=c/a, \quad \beta_1=b+g, \quad \beta_2=b-g,$$

Lemma 1. Equation (8) with initial condition (2) is equivalent to the system of integral equations (SIE):

$$\vartheta(t,x)=\frac{1}{2}\varphi_1(q(0,t,x))+\frac{1}{2}\beta_1 u+\frac{\beta_2}{2}\int_0^t\vartheta(s,q)ds+\int_0^t F(s,q,u(s,q))ds, \quad (10)$$

$$u(t,x)=u_0(p(0,t,x))+\int_0^t\vartheta(s,p(s,t,x))ds, \quad (11)$$

where

$$[2\vartheta(t,x)-\beta_1 u(t,x)]\big|_{t=0}=\varphi_1(x).$$

Proof of Lemma 1. Let $\vartheta(t,x)$, $u(t,x)$ - solution SIE (10)-(11).

Finding partial derivatives up to the second order from the integral equation (10) and partial derivatives of the first order from (11), we obtain equation (8) and notation (9). The system satisfies the initial condition (2).

Now, with the help of MAA from (3), (4) with (2), we must obtain (10), (11), for this we write equation (3) in a form that is convenient for using the indicated method:

$$\frac{\partial z}{\partial t}+a\frac{\partial z}{\partial x}=\beta_2\vartheta(t,x)+2F(t,x,u), \quad (12)$$

where

$$z(t,x)=2\vartheta(t,x)-\beta_1 u(t,x).$$

From (12) with the initial condition (2) using MAA, we obtain:

$$2\vartheta(t,x)-\beta_1 u=\varphi_1(q(0,t,x))+\beta_2\int_0^t\vartheta(s,q)ds+2\int_0^t F(s,q,u(s,q))ds,$$

Therefore, we obtain (10). The validity of (11) follows from the notation (9) with the help of MDA. We have proved Lemma 1.

Further, in (10), substituting (11), we obtain an integral equation with respect to $\vartheta(t,x)$.

$$\vartheta(t,x)=A(t,x;\vartheta)\equiv\frac{1}{2}\varphi_1(q(0,t,x))+\frac{1}{2}\beta_1\left(u_0(p(0,t,x))+\int_0^t\vartheta(s,p(s,t,x))ds\right)+$$
$$+\frac{\beta_2}{2}\int_0^t\vartheta(s,q)ds+\int_0^t F\left(s,q,\left(u_0(p(0,s,q))+\int_0^s\vartheta(v,p(v,s,q))dv\right)\right)ds. \quad (13)$$

Lemma 2. There exists $T^*>0$, such that the integral equation (13) has a unique solution in $\overline{C}(G_2(T^*))$.

Proof of Lemma 2.

For equation (13), we apply the principle of compressed mappings.

Let in the region $G_2(T)$ at $T<T_*$:

$$\|\vartheta-\phi\|\leq M,$$

where

$$\phi(x)=\frac{1}{2}\varphi_1(q(0,t,x))+\frac{1}{2}\beta_1u_0(p(0,t,x))+\int_0^t F\left(s,q,\left(u_0(p(0,s,q))+\int_0^s\vartheta(\nu,p(\nu,s,q))d\nu\right)\right)ds+$$

$$+\int_0^t\omega_0(p(0,s,q))ds+\int_0^t\int_0^\tau\omega_1(q(0,s,p(s,\tau,q))dsd\tau+\int_0^t\int_0^\rho\int_0^\nu\int_0^\tau K(\tau,s)u(s,q(\tau,\nu,p(\nu,\rho,q)))dsd\tau d\nu d\rho.$$

Fair ratings:

$$\|A\vartheta-\phi\|\leq bKT+(\gamma K+\|f\|)\frac{T^2}{2}=\Omega_0(T),$$

$$\left\|A_1\vartheta^1-A_1\vartheta^2\right\|\leq\Omega_1(T)\left\|\vartheta^1-\vartheta^2\right\|,$$

where

$$\|\vartheta\|\leq\|\phi\|+M=K,$$

$$\Omega_1(T)=bT+(\gamma+\|f\|)\frac{T^2}{2}.$$

Let T_0,T_1- positive roots of the equations respectively:

$$\Omega_0(T)=M,\ \Omega_1(T)=1.$$

It follows from this that the operator A under $T<T^*=\min\{T_0,T_1\}$ performs a compressed mapping. Therefore, the stated initial problem has a unique solution. This solution can be obtained by the method of successive approximations.

We found $\vartheta(t,x)$. Substituting it into (11), we find the unknown function $u(t,x)$. And the unknown function $\omega(t,x)$ is determined from (7).

Conclusion:

In this work we do not reduce the initial value problem (1)-(2) to a canonical form, we reduce it to a system of integral equations. The developed scheme of using the method of additional argument for partial integro-differential equation of fourth order can be applied to partial integro-differential equations of higher order of other classes.

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