GENERATORS AND RELATIONSHIPS IN GENERALIZED M-TRIANGULAR GROUPS OVER AN ASSOCIATE RING. I

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Abstract. *The question of representing linear groups (and related constructions) by generating elements and defining relations has always been of interest in general combinatorial group theory. Today, a large amount of magazine and book materials have already accumulated in this direction. New research methods also emerged. One of them is the universal combinatorial transformation method, the essence of which is to transform words of the selected generative alphabet of the group under study to their standard forms. The paper provides a description through generators and defining relations of generalized m-triangular groups* $T_{n,m}^{\circ}(R), n \geq 2$ *defined over an arbitrary non-zero associative ring. Based on this result, combinatorial descriptions of the projective factors of the named groups* $PT_{n,m}^{\circ}(R)$ *are also found. The solution to these problems is based on the mentioned transformation method.*

Keywords: *generators, relations, quasi-multiplication, quasigroup, generalized m-triangular group, standard forms, transformation of letters, completeness of relations, projective factor.*

INTRODUCTION

The representation of linear (and close to them) groups in terms of generators and relations is one of the main issues in combinatorial group theory. This section has long grown into a special direction in general theory and is currently experiencing rapid development. Within the framework of this topic, we can note the remarkable (and already classic) results [1]–[4]. The proposed work is also devoted to the named section, or rather, here we will give a combinatorial description of generalized m-triangular groups of degree $n \geq 2$ over an arbitrary associative ring.

Throughout, we assume an arbitrary nonzero associative ring for which the existence of 1 is not necessary. Through $\circ$, as always, we denote quasi-multiplication in R,т.е. $x \circ y = x + xy + y$ for elements $x,y \in R$. Element x from is called quasi-invertible if for it $x \circ y = 0 = y \circ x$ at some $y \in R$. Given a quasi-reversible, its quasi-inverse is always determined uniquely and it is denoted as $y = x'$. The set of all quasi-invertible elements R° from R is non-empty (for example $0 \in R^{\circ}$) and it forms a group relative to the operation $\circ$. Unit in R° element 0 will serve. We call this group the quasigroup of the ring.

In the special case, putting instead the ring of (upper) triangular matrices $T_n(R)$, we come to the concept of a generalized triangular group $[T_n(R)]^o = T_n^o(R)$ degrees n above the ring R. For natural $m, 1 \le m \le n$, by analogy with [5] (see p. 24) we denote by $T_{n,m}^o(R)$ set of matrices from $T_n^o(R)$ from m-1 zero diagonals above the main one, i.e.

$$T_{n,m}^o(R) = \{x = (x_{ij}) \in T_n^o(R) : o < j - i < m \to x_{ij} = 0\}.$$

Let us show that the introduced sets form subgroups $T_{n,m}^o(R)$. To do this, we just need to check the closedness $T_{n,m}^o(R)$ with respect to matrix quasi-multiplication and the operation of taking a quasi-inverse element. Let, along with the above $x = (x_{ij})$ put from $T_{n,m}^o(R)$ another matrix $y = (y_{ij})$. As is easy to see, for positions $\langle i, j \rangle,\ 0 < j - i < m,$ quasi-products of these matrices satisfy the formulas $(x \circ y)_{ij} = x_{ij} + \sum_{1 \le x \le n} x_{ik} y_{kj} + y_{ij} = \sum_{1 \le x \le n} x_{ik} y_{kj}$.

Since when $k \ne i\ 0 < j - i < m\ \& i \le k \le j \to 0 < k - i < m$, first factors x_{ik}, $i < k \le j$, the last amount will be equal to zero. When $k = i$ we have $y_{kj} = 0$. Thus, the equalities $(x \circ y)_{ij} = 0$ true for all the above positions $\langle i, j \rangle$, those isolation in $T_{n,m}^o(R)$ occurs.

To continue our reasoning further, we need the following notation: for $\varepsilon \in R^\circ\ d_i(\varepsilon) -$ matrix of $T_n^o(R)$, differing from the zero matrix only by position $\langle i, i \rangle$, where is the element ε; the same way $t_{ij}(\lambda),\ i \ne j$, will mean matrix (also from $T_n^o(R)$), obtained from the zero matrix by replacing its position $\langle i, j \rangle$ for argument $\lambda \in R$ (they are called quasi-transvections). For the introduced matrices the formulas are obvious: $d_i^/(\varepsilon) = d_i(\varepsilon^/),\ t_{ij}^/(\lambda) = t_{ij}(-\lambda).$

$(^/)$

Just now x– arbitrary matrix of $T_{n,m}^o(R)$. From equality (*sf*) of this work (see paragraph I) it follows that $x^/ = f_1^/ \circ \ldots \circ f_{n-m}^/ \circ d_n^/(\varepsilon_n) \circ \ldots \circ d_1^/(\varepsilon_1)$

(f_i – some words composed of quasi-products of transvections of the form $t_{ik}(\lambda_k)$). Application to the right side of the last equality of relations ($^/$) will lead us to a representation of the matrix x/ consisting of a quasi-product of (a finite number of) diagonal letters $d_k(\varepsilon)$ and quasi-transvections $t_{ij}(*)$. And this, according to the closedness already established above, means belonging to $T_{n,m}^o(R)$ not only x, but also its quasi-inverse matrix $x^/$. So, group inclusion $T_{n,m}^o(R) \le T_n^o(R)$ we have completely installed it. Entered group $T_{n,m}^o(R)$ we will call the generalized m-triangular group of degree $n \ge 2$ over the ring R. As noted above, our main goal in this part of the work is to define in terms of generators and relations of triangular groups $T_{n,m}^o(R)\ m = 1,2,\ldots,n$. It is carried out exactly the same for all specified values. m. Entered groups in $T_n^o(R)$ form a descending chain

$$T_n^o(R) = T_{n,1}^o(R) > T_{n,2}^o(R) > \ldots > T_{n,n}^o(R) = D_n^o(R)$$

(where $D_n^0(R) \simeq R^0 \underset{n\ раз}{\times \ldots \times} R^0$ – diagonal in $T_n^o(R)$). A similar serial description was carried out earlier in [6] for subgroups of the complete linear group $GL_n(\Lambda), n \ge 2,$ over the local ring Λ (with small restrictions on Λ), containing a group of diagonal matrices $D_n(\Lambda)$. In concept, our work is also close to work [7], where the combinatorial structure of a triangular group of any (even infinite) order was studied.

It is easy to see that if there is a 1 in R, the mapping

$$T_{n,m}(R) \to T^o_{n,m}(R),\ e + x \to x$$

(e– unit order matrix n), defines an isomorphism of groups. Therefore, the groups introduced above $T^o_{n,m}(R)$ are generalizations of the usual m-triangular groups (respectively) to the most general cases of associative rings R. When solving the problem, we again use the transformation method developed in [9] and [10].

Standart forms in $T^o_{n,m}(R)$

They are defined relative to some generating system of the named group. As such we take the system

$$d_k(\varepsilon), \varepsilon \in \mathrm{R}^o,\ 1 \le k \le n;\ t_{ij}(\lambda), \lambda \in R,\ j\text{-}i \ge m. \qquad (g)$$

The fact that the group $T^o_{n,m}(R)$ is generated by the alphabet (g), follows directly from Theorem 1 of this paper. Under the step forms i here we understand words of the form $f_i = \prod_{i+m \le x \le n} t_{ik}(\lambda_k)$ (where multiplication is quasi-multiplication and the order of the factors is unimportant). As standard forms, all possible combinations of the alphabet (g) of the form are declared here $x = d_1(\varepsilon_1) \circ \ldots \circ d_n(\varepsilon_n) \circ f_{n-m} \circ \ldots \circ f_1$ (*sf*)

(where m=n expression $f_{n-m} \circ \ldots \circ f_1$ meaning is given 0).

Regarding the entered forms, it occurs

Theorem 1. Any matrix x from $T^o_{n,m}(R)$, $n \ge 2$, presented in standard form (*sf*), and such a representation is unique.

Proof. Uniqueness. Just $f_1 = t_{1,m+1}(\lambda_{m+1}) \circ \ldots \circ t_{1n}(\lambda_n)$. Here $d_2(\varepsilon_2) \circ \ldots \circ d_n(\varepsilon_n) \circ f_{n-m} \circ \ldots \circ f_2$ has a cell-diagonal appearance $diag(0, x_1)$, $d_1(\varepsilon_1) \circ f_1$ has the same first row as x, i.e.. $x_{11}, 0, \ldots, 0, x_{1,m+1}, \ldots, x_{1n}$. Equating the corresponding positions here gives us $\varepsilon_1 = x_{11}, \lambda_{m+1} = \varepsilon_1^/ x_{1,m+1} + x_{1,m+1}, \ldots, \lambda_n = \varepsilon_1^/ x_{1n} + x_{1n}$, т.е. ε_1 и f_1 matrice x are determined unambiguously. Moving now from x to the matrix $d_1^/(\varepsilon_1) \circ x \circ f_1^/ = d_2(\varepsilon_2) \circ \ldots \circ d_n(\varepsilon_n) \circ f_{n-m} \circ \ldots \circ f_2$, we similarly conclude the uniqueness ε_2 and f_2. The process described on (n-m)-м step leads us to the conclusion about the uniqueness ε_{n-m} and f_{n-m}. And then the equalities $\varepsilon_k = x_{kk}, n - m < k \le n$, already take place in an obvious way.

As for the *existence part of the theorem*, it is a direct consequence of Theorem 3 of this paper. Therefore, we can omit it here too. The case m=n can also be included in this theorem, if we assume that there $f_{n-m} \circ \ldots \circ f_1 = 0$.

Constitutive relations

In the alphabet (g) we can write the following (directly verifiable) group relations $T^o_{n,m}(R)$:

1. $d_i(\varepsilon) \circ d_i(\sigma) = d_i(\varepsilon \circ \sigma)$;
2. $d_i(\varepsilon) \circ d_k(\sigma) = d_k(\sigma) \circ d_i(\varepsilon), \quad i \ne k$;

3. $t_{ik}(\lambda)\circ t_{ik}(\alpha)=t_{ik}(\lambda+\alpha)$;

4. $t_{ik}(\lambda)\circ t_{rj}(\alpha)=t_{rj}(\lambda)\circ t_{ik}(\alpha),\ k\neq r,\ i\neq j$;

5. $t_{ik}(\lambda)\circ t_{kj}(\alpha)=t_{ij}(\lambda\alpha)\circ t_{kj}(\alpha)\circ t_{ik}(\lambda)$;

6. $t_{ik}(\lambda)\circ d_i(\varepsilon)=d_i(\varepsilon)\circ t_{ik}(\lambda+\varepsilon^{/}\lambda)$;

7. $t_{ik}(\lambda)\circ d_k(\varepsilon)=d_k(\varepsilon)\circ t_{ik}(\lambda+\lambda\varepsilon)$;

8. $t_{ik}(\lambda)\circ d_r(\varepsilon)=d_r(\varepsilon)\circ t_{ik}(\lambda);\ r\neq i,k.$

Our immediate goal is to show the completeness of the system of relations 1–8 for the group $T^{o}_{n,m}(R)$ in generating (g). For this purpose, we introduce (binary) relations on the set of all words of the alphabet (g) $\overset{i}{\rightarrow}$, $1\leq i\leq n-m$, put in $W\overset{i}{\rightarrow}V$ if and only if the words W and V related by the relation $W=X\circ V$, where X– some word that does not contain non-zero quasi-transvections of the form $t_{kj}(*),\ k\leq i$. How to easily check entered relationships $\overset{i}{\rightarrow}$ are reflexive and transitive.

Next, we will need the following

Theorem 2 (about the transformation of letters). Let f_i– some form of step i and x– non-zero letter of the alphabet (g), for which $x=t_{pq}(\lambda)$ condition is considered fulfilled $p\geq i$. Then for them, using relations 3–8, you can perform the transformation $V=f_i\circ x\overset{i}{\rightarrow}g_i$, where g_i– also some form of stage i.

The *proof* is combinatorial and is carried out in two stages. Below we, to simplify the entries under $f_i(\neq r)$ let's agree to understand the form f_i, not containing a letter of the form $t_{ir}(*),\ *\neq 0$.

Stage I. $x=d_k(\varepsilon)$

Here, using relations 6–8, we will have $V=f_i(\neq n)\circ[t_{in}(\lambda)\circ d_k(\varepsilon)]=[f_i(\neq n)\circ d_k(\varepsilon)]\circ t_{in}(\lambda)$. Continuing this movement $d_k(\varepsilon)$ and further, we arrive at the required form like this $V=d_k(\varepsilon)\circ t_{i,i+m}(*_{i+m})\circ\ldots\circ t_{in}(*_n)\rightarrow t_{i.i+m}(*_{i+m})\circ\ldots\circ t_{in}(*_n)=g_i$.

Stage II. $x=t_{rj}(\lambda)$.

Here our consideration branches out as follows.

a)r=i. Applying relations 4 and 3, here we obtain the required form as follows $V=f_i(\neq j)\circ[t_{ij}(*)\circ t_{ij}(\lambda)]=[f_i(\neq j)\circ t_{ij}(*+\lambda)]=g_i$.

в) *r>i*. In this case, using relations 4 and 5, we will have $V=f_i(\neq r)\circ[t_{ir}(*)\circ t_{rj}(\lambda)]=[f_i(\neq r)\circ t_{rj}(\lambda)]\circ t_{rj}(\alpha)\circ t_{ir}(*)=t_{rj}(*)\circ f_i(\neq r)\circ t_{ij}(*)\circ t_{ir}(*)\rightarrow$.
$[f_i(\neq r)\circ t_{ij}(*)]\circ t_{ir}(*)$.

The resulting word by applying the already analyzed point a) to the selected segment leads us to the required form as $V\overset{i}{\rightarrow}f_i(\neq r)\circ t_{ir}(*)=g_i$. Theorem 2 is proven.

3. Group View $T^{o}_{n,m}(R)$

We are now ready to formulate a basic statement about the representation of the named group.

Theorem 3. Generalized *m*- triangular group $T^o_{n,m}(R)$, $n \geq 2$ $(1 \leq m \leq n)$, over the associative ring $R \neq \{0\}$ in generators (g) is represented by relations 1–8.

The proof consists of two parts.

I. Reduction to standard form.

In this part we will show the reducibility of any word W of the alphabet (g) to its standard form S(W) using relations 1–8. Without loss of generality, a given word can be considered represented in the form

$$W \xrightarrow{i} f_1 \circ X, \qquad (0)$$

Where f_1– some form of stage 1 and X is its corresponding complement. Let further, $X = x \circ X_1$ x– the first letter of the complement X. Applying transformation theorem 2 (i.e. using relations 3–8), we reduce the given word to the form $W = [f_1 \circ x] \circ X_1 \xrightarrow{1} g_1 \circ X_1$,. we get a notation of the same form (0), but with a shortened complement X. Continuing this reduction further (until all X is exhausted), we come to a notation of the form $W \xrightarrow{1} f_1$ (where f_1– another form of step 1).Last according to definition $\xrightarrow{1}$ means that $W = Y_1 \circ f_1$, where is the (already left) complement Y_1 does not contain quasi-transvection $t_{1j}(*), * \neq 0$. Now we do the same with Y_1 and extract the shape from it f_2(ступени 2), we have $W = Y_2 \circ f_2 \circ f_1$, where is the complement Y_2 does not contain a quasi-transvection of the form $t_{ik}(*), * \neq 0,\ i \leq 2,$ etc. The described process of form splitting off at the (n-m)th step leads us to the notation

$$W = Y_{n-m} \circ f_{n-m} \circ \ldots \circ f_2 \circ f_1,$$

where is the word already Y_{n-m} (a-priory $\xrightarrow{n-m}$) does not contain transvection species $t_{ik}(*), * \neq 0,\ i \leq n-m,$ those. it consists entirely of diagonal letters of the alphabet (g). By applying relations 1 and 2 to it, it is now reduced to the form $d_1(\varepsilon_1) \circ \ldots \circ d_n(\varepsilon_n)$ in an obvious way, i.e. the given word is reduced to its standard form S(W).

II.Completeness of relations 1–8.

Let now W=0 arbitrary group relation $T^o_{n,m}(R)$(in generators (g)). Having written the left-hand side in its standard form (using relations 1–8), we replace it with *S*(*W*)=0. But according to Theorem 1, the latter is possible only for zero letters of the form S(W). And this already means that the given relation W=0 can be derived from 1–8. Theorem 3 is completely proven.

As we noted above, when *m*=*n* is $f_{n-m} \circ \ldots \circ f_1 = 0$, those. in this case, both the transvections from (g) and the (related to us) relations 3–8 disappear from our field of consideration. In other words (g) is replaced with a subalphabet $d_i(\varepsilon)$, $\varepsilon \in R^o$, $1 \leq i \leq n,$ and the relations 1 - 8 with 1.2, and here Theorem 3 simply turns into Dikov's definition of the diagonal subgroup $D^0_n(R)$.

4.Assignment of the projective factor $PT^o_{n,m}(R)$.

Based on (main) Theorem 3, in this section we give a combinatorial representation of the group factor $T^o_{n,m}(R)$ in the centre $C = centT^o_{n,m}(R)$. And to do this, we first need to calculate this center, or rather, find some C-generating system of words W of the alphabet (g). Then the factor under consideration will be presented as $T^o_{n,m}(R) = \langle (g) \,\|\, 1-8 \,\&\, W = 0 \rangle$ (см.[11], стр.77).

In the case where m=n, the group being studied $T^o_{n,m}(R)$ turns into a (classical) diagonal group $D^0_n(R)$. Setting its projective factor is not difficult and it is not interesting.

Cases *m*<*n*в $T^o_{n,m}(R)$ require additional research. Let $x = (x_{ij})-$ an arbitrary matrix from the center C. Taking also an arbitrary (diagonal) matrix $d_k(\varepsilon),\ \varepsilon \in R^o,\ 1 \le k \le n,$ we have

$$d_k(\varepsilon) \circ x = x \circ d_k(\varepsilon).$$

The latter will obviously lead us $\varepsilon \circ x_{kk} = x_{kk} \circ \varepsilon,$ those to switch on

$$x_{kk} \in centR^o. \qquad (\in)$$

Let's consider in *x* its "corner" positions x_{ij} (i.e., positions for which $i \le n-m$ и $j \ge i+m$). For these elements we also have the equalities $t_{ij}(\lambda) \circ x = x \circ t_{ij}(\lambda)$

(λ–an arbitrary element from R). Comparison in last positions $< i,i >,\ < j,j >,\ < j,i >$ will lead us to $\lambda x_{ij} = 0 = x_{ij}\lambda$ and

$$x_{ii}\lambda = \lambda x_{jj}. \qquad (s)$$

Thus, in the central matrix, all its corner elements x_{ij} are required to enter the annulment *AnnR*, and its diagonal elements (in addition to inclusions) must also satisfy the requirements of "scalarity" (s). Now check that the matrix x, satisfying all the above conditions, will be central in $T^o_{n,m}(R)$, is no longer difficult. It also became obvious that the center C is generated by quasi-transvections $t_{ij}(\delta),\ \delta \in AnnR\ (i \le n-m,\ j \ge i+m$, and all "scalar" words $d_1(\varepsilon_1) \circ \ldots \circ d_n(\varepsilon_n)$.

Summarizing these facts, we can formulate the following result.

Theorem 4. Projective generalized m-triangular group $PT^o_{n,m}(R),\ n \ge 2\ (1 \le m < n),$ over the associative ring $R \ne \{o\}$ in generators (g) is represented by relations 1–8, angular relations $t_{ij}(\delta) = 0,\ \delta \in AnnR\ (i \le n-m,\ j \ge i+m),$ and with the following "scalar" relations $d_1(\varepsilon_1) \circ \ldots \circ d_n(\varepsilon_n) = 0$
$(\varepsilon_k \in centR^o).$

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