GENERATORS AND RELATIONSHIPS IN GENERALIZED M-TRIANGULAR GROUPS OVER AN ASSOCIATE RING.II

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Abstract. *This work continues the research begun in its first part, where generalized m-triangular groups* $T^o_{n,m}(R),\ n\geq 2,\ (1\leq m\leq n),$ *over an arbitrary associative ring R were studied from the standpoint of generators and relations. The named generators and relations were identified there uniformly for all values of m. A combinatorial description of the projective factors of the* $PT^o_{n,m}(R)$ *named groups was also found there. In combinatorial theory, descriptions of not only some classical subgroups of a complete linear group, but also their natural parts are of interest. In this part of the work, the generators and defining relations of the generalized elementary triangular group* $ET^o_{n,m}(R)$ *and its projective factor* $PET^o_{n,m}(R)$ $(n\geq 2,\ 1\leq m\leq n)$ *are similarly identified over an arbitrary associative ring R.*
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In this work we continue the research begun in its first part. Therefore, keeping all the definitions and notation as before, here too we consider R to be an arbitrary associative non-zero ring. Let us additionally accept the following notation: for numbers $i,k,\ 1\leq i<k\leq n,$ and an argument

$$\varepsilon\in R^o\ \ d_{ik}(\varepsilon)=d_i(\varepsilon)\circ d_k(\varepsilon^{/});\ [x,y]=x^{/}\circ y^{/}\circ x\circ y-$$

element switch $x,y\in R^o$, $[R^o,R^o]-$ group switch R^o, those subgroup in R^o, generated by all its commutators (it forms in R^o, normal subgroup);

$$ET^o_{n,m}(R)=\left\langle d_{ik}(\varepsilon),\ \varepsilon\in R^o,\ 1\leq i<k\leq n;\ t_{ik}(\lambda),\ \lambda\in R,\ m<i+m\leq j\leq n\right\rangle,$$

$ET^o_{n,m}(R)$–subgroup im $T^o_{n,m}(R)$, generated by all the (elementary) matrices indicated there. This group $ET^o_{n,m}(R)$ we will call the generalized elementary *m*-triangular group of degree *n* above the ring *R* (corresponding to the m-th diagonal). Our goal in this part of the work is to present groups $ET^o_{n,m}(R),\ n\geq 2,\ (1\leq m\leq n),$ in terms of generators and relations. Our representation is produced in exactly the same way (i.e. serially) for all specified values of m and also uses the transformation method developed in [1]–[5]. Despite their apparent closeness, the group considered here $ET^o_{n,m}(R)$ has significant differences from its predecessors $T^o_{n,m}(R)$.

1. Standard forms in $ET^{o}_{n,m}(R)$

As the expansions show $d_1([\varepsilon,\sigma]) = d^{/}_{1k}(\sigma\circ\varepsilon)\circ d_{1k}(\varepsilon)\circ d_{1k}(\sigma)$ и $d_k([\varepsilon^{/},\sigma^{/}]) = d^{/}_{1k}(\varepsilon\circ\sigma)\circ d_{1k}(\varepsilon)\circ d_{1k}(\sigma),\ 1<k\le n,$ single matrices $d_j(\tau),\ 1\le j\le n,$ c an argument $\tau\in[R^{o},R^{o}]$ are some elements from $ET^{o}_{n,m}(R)$. To represent a group $ET^{o}_{n,m}(R)$ we will choose not the system that generates it, but a more symmetrical alphabet $d_{ik}(\varepsilon),\ \varepsilon\in R^{o},\ 1\le i<k\le n;\ d_q(\sigma),\ \sigma\in[R^{o},R^{o}],\ 1\le q\le n;$

$$t_{ij}(\lambda),\ \lambda\in R,\ m<i+m\le j\le n. \qquad \text{(Eg)}$$

And in this part, we use standard forms of elements from $ET^{o}_{n,m}(R)$. The forms of stage *i* and here we define as $f_i = \prod t_{ik}(\lambda_k)$, where *k* runs through a lot $\{i+m,\dots,n\}$ (in no particular order). As standard forms, we here declare all possible combinations of the alphabet (Eg) of the form $d_{1n}(\varepsilon_1)\circ\dots\circ d_{n-1,n}(\varepsilon_{n-1})\circ d_n(\varepsilon)\circ f_{n-m}\circ\dots\circ f_1,$ *(sf)*

Regarding the entered forms, it occurs

Theorem 1. Any matrix from x $ET^{o}_{n,m}(R),\ n\ge 2,\ (1\le m\le n),$ is represented in standard form (sf), and in a unique way.

The proof of this theorem is carried out without significant changes as in the corresponding theorem from the first part. We will not reproduce it here. In order to maintain uniformity in reasoning, here too we *m*=*n* в (*sf*) будем считать $f_{n-m}\circ\dots\circ f_1 = 0$.

2. System of defining relations

Let us write in the alphabet (Eg) the following (easily verifiable) relations of the group $ET^{o}_{n,m}(R)$:

1. $d_{ik}(\varepsilon) = d_{in}(\varepsilon)\circ d_{kn}(\varepsilon^{/}),\ \ \mathrm{k<n};$
2. $d_q(\sigma) = d_{qn}(\sigma)\circ d_n(\sigma)\ \ 1\le q<n;$
3. $d_n(\sigma)\circ d_n(\varepsilon) = d_n(\sigma\circ\varepsilon);$
4. $d_{in}(\varepsilon)\circ d_{in}(\sigma) = d_{in}(\varepsilon\circ\sigma)\circ d_n([\varepsilon^{/},\sigma^{/}]);$
5. $d_{kn}(\varepsilon)\circ d_{in}(\sigma) = d_{in}(\sigma)\circ d_{kn}(\varepsilon)\circ d_n([\varepsilon^{/},\sigma^{/}]),\ k>i;$
6. $d_n(\sigma)\circ d_{in}(\varepsilon) = d_{in}(\varepsilon)\circ d_n(\varepsilon\circ\sigma\circ\varepsilon^{/});$
7. $t_{in}(\lambda)\circ d_n(\varepsilon) = d_n(\varepsilon)\circ t_{in}(\lambda+\lambda\varepsilon);$
8. $t_{ik}(\lambda)\circ d_n(\varepsilon) = d_n(\varepsilon)\circ t_{ik}(\lambda),\ k<n;$
9. $t_{ik}(\lambda)\circ d_{kn}(\varepsilon) = d_{kn}(\varepsilon)\circ t_{ik}(\lambda+\lambda\varepsilon);$
10. $t_{ik}(\lambda)\circ d_{in}(\varepsilon) = d_{in}(\varepsilon)\circ t_{ik}(\varepsilon^{/}\lambda+\lambda),\ k<n;$
11. $t_{in}(\lambda)\circ d_{in}(\varepsilon) = d_{in}(\varepsilon)\circ t_{in}(\lambda+\lambda\varepsilon^{/}+\varepsilon^{/}(\lambda+\lambda\varepsilon^{/}));$
12. $t_{in}(\lambda)\circ d_{rn}(\varepsilon) = d_{rn}(\varepsilon)\circ t_{in}(\lambda+\lambda\varepsilon^{/});$
13. $t_{ik}(\lambda)\circ d_{rn}(\varepsilon) = d_{rn}(\varepsilon)\circ t_{ik}(\lambda),\ k<n,\ r\ne i,k;$
14. $t_{ik}(\lambda)\circ t_{ik}(\alpha) = t_{ik}(\lambda+\alpha);$
15. $t_{ik}(\lambda)\circ t_{kj}(\alpha) = t_{ij}(\lambda\alpha)\circ t_{kj}(\alpha)\circ t_{ik}(\lambda);$

16. $t_{ik}(\lambda)\circ t_{rj}(\alpha)=t_{rj}(\alpha)\circ t_{ik}(\lambda),\ i\neq j,\ k\neq r.$

To continue further reasoning, we introduce on the set of all words of the alphabet (*Eg*) contact $\overset{i}{\rightarrow}$, $1\leq i\leq n-m$, put in $W\overset{i}{\rightarrow}V$ if and only if the words *W*и*V* are related to each other by the relationship $W=XV$, where *X* does not contain non-zero transvections $t_{kj}(*),\ k\leq i.$ These relations are reflexive and transitive.

And here the auxiliary (transformational on the right) is correct

Theorem 1. Just f_i– some form of step $i\,(1\leq i\leq n-m)$ and *x*– non-zero letter of the alphabet (*Eg*), for which at $x=t_{Rq}(\lambda)$ inequality is considered satisfied $p\geq i$. Then for them, using relations 7–16, you can perform transformations $V=f_i x\overset{i}{\rightarrow}g_i$, where g_i– also some (already different!) form of stage *i*.

The proof is combinatorial and distinguishes the following cases.I. *x*– diagonal letter

Here we apply relations 7–13, 16 (and understand by $f_1(\neq r)$ form without letters $t_{ir}(\lambda),\ \lambda\neq 0)$, we have $V=f_1(\neq r)\circ[t_{1r}(*)\circ x]=[f_1(\neq r)\circ x]\circ t_{1r}(*_r)$. Continuing this movement x and further, we arrive at the required form like this $V=(x\circ g_1)\overset{i}{\rightarrow}g_1$.

II. $x=t_{rj}(\lambda)$

In this case the transformation $V=f_1\circ x\overset{i}{\rightarrow}g_1$ is carried out using relations 14–16 as in Theorem 2 from the first part. We also omit these repeated details here.

Left transformational transformations

This point is also auxiliary. Here we need a diagonal subalphabet

$$d_{ik}(\varepsilon),\ \varepsilon\in R^o,\ 1\leq i<k\leq n,\ d_j(\sigma),\ \sigma\in[R^o,R^o],\ 1\leq j\leq n. \qquad (Ed)$$

Put in $V\overset{i}{\leftarrow}W$ if and only if these words are related by the relation *V=WY*, where the word Y does not contain non-zero letters of the form $d_{kj}(\varepsilon)\ (\varepsilon\neq 0),\ k\leq i.$ These relations are also reflexive and transitive.

Below we also need the following

Theorem 3 (on transformation on the left).

Using relations 1–6, any word *V* of the alphabet (Ed) can be written in the form

$$d_{1n}(\varepsilon_1)\circ\ldots\circ d_{n-1,n}(\varepsilon_{n-1})\circ d_n(\varepsilon). \qquad (d)$$

Prove. Without loss of generality, the word in question can be considered represented in the form $V=Y\circ d_{1n}(*)$. Applying relations 1 and 2 to part *Y, V* can be considered to consist only of letters of the form $d_{1n}(\varepsilon)$ и $d_n(\sigma)$. Just now $Y=Y_1\circ y$ those *y*–the latest letter in *Y*. Applying relations 4–6, we will then have $V=Y_1\circ[y\circ d_{1n}(*)]\overset{i}{\leftarrow}Y_1\circ d_{1n}(\varepsilon)$, those With this operation we have achieved a reduction in the length of Y. Continuing these reductions further, we arrive at the notation $V\overset{i}{\leftarrow}d_{1n}(\varepsilon_1)$. This is by definition a relationship. $\overset{i}{\leftarrow}$ means that $V=d_{1n}(\varepsilon_1)\circ X_1$, where *X*– some word of the alphabet (*Ed*), not containing non-zero letters of the form $d_{1n}(*)\ (*\neq 0)$. Similarly, splitting off from the letter *X* $d_{2n}(\varepsilon_2)$, we'll have $V=d_{1n}(\varepsilon_1)\circ d_{2n}(\varepsilon_2)\circ X_1$, where X_1

does not contain letters of the form $d_{in}(*),\ * \neq 0,\ i \leq 2,$ The Described process of cleavages on (n-1)-м step leads us to decomposition $V = d_{1n}(\varepsilon_1) \circ \ldots \circ d_{n-1,n}(\varepsilon_{n-1}) \circ X_{n-2}$, where X_{n-2} does not contain letters of the form $d_{kn}(*),\ * \neq 0,\ k < n,$ consists entirely of single letters $d_n(*)$. By applying relations 3, the latter is now reduced to the form $d_n(\varepsilon)$ in an obvious way. The theorem has been proven.

3. Combinatorial group assignment $ET^{o}_{n,m}(R)$

The main result of the work will be formulated as

Theorem 4. Generalized elementary m- triangular group $ET^{o}_{n,m}(R),\ n \geq 2,\ (1 \leq m \leq n),$ over the associative ring $R \neq \{0\}$ in the generators (Ed) is given by the relations 1–16.

The proof here is divided into two parts.

Reduction to standard form

In this section we will show that using relations 1–16, every word W alphabit (Ed) can convert to its standard form $s(W)$. As in Theorem 3, assuming W composed only of letters $t_{ik}(\lambda),\ d_{1n}(\varepsilon),\ d_n(\sigma)$ and repeating with minor changes the reasoning of Theorem 4 from the first part of the work (i.e., the relations 1,2 and 7–16), word W can be written in the form

$$W = D \circ f_{n-m} \circ \ldots \circ f_2 \circ f_1,$$

where D– some subalphabet word (Ed). Applying now to D just proved Theorem 3 (i.e. relations 1–6), we reduce it to the form $D = d_{1n}(\varepsilon_1) \circ \ldots \circ d_{n-1,n}(\varepsilon_{n-1}) \circ d_n(\varepsilon)$. So the equality W=$s(W)$ from relations 1–16 can indeed be extracted.

II.Completeness of relations.

Let it now W=0– arbitrary group ratio $ET^{o}_{n,m}(R)$ in generating (Ed). Applying the result of step I to its left side, we replace the latter with $s(W)$=0. But according to Theorem 1, the latter is possible only with zero letters of the word $s(W)$. And this already means that the relation W=0 can be derived from 1–16. Theorem 4 is completely proven.

5.Description of the projective factor $PET^{o}_{n,m}(R)$

Starting from the main Theorem 4, we present here a combinatorial definition of the factor $PET^{o}_{n,m}(R) = ET^{o}_{n,m}(R)/C$ elementary group $ET^{o}_{n,m}(R)$ at its center $C = centET^{o}_{n,m}(R)$. Happens m=n here is trivial and is of no interest to us.

Counting everywhere below m<n, consider in $ET^{o}_{n,m}(R)$ arbitrary central matrix x=(x_{ij}) (i.e., a matrix from C). Taking arbitrarily the matrix $d_{in}(\varepsilon),\ \varepsilon \in R^{o},\ 1 \leq i < n,$ we have equality $d_{in}(\varepsilon) \circ x = x \circ d_{in}(\varepsilon)$.

The latter gives us that

$$\varepsilon \circ x_{ii} = x_{ii} \circ \varepsilon,\ 1 \leq i < n, \qquad (i)$$

and $\varepsilon^{/} \circ x_{nn} = x_{nn} \circ \varepsilon^{/} (\rightarrow \varepsilon \circ x_{nn} = x_{nn} \circ \varepsilon)$, those. equality (i) is true for all i=1, 2,…,n. Thus, inclusions $x_{kk} \in centR^{o},\ k = 1,2,\ldots,n,$

take place here too. Next, inclusions $x_{ij} \in AnnR$ and equality

$$x_{ii}\lambda = \lambda x_{jj} \qquad (s)$$

$(i \le n-m,\ j \ge i+m)$ are correct and they are shown as in the first part of the work.

Thus, in the central matrix x all its angular positions x_{ij} must be included in the annulment *AnnR*, and its diagonal elements, along with c(∈) must also satisfy the "scalarity" conditions (s). And the fact that the matrix $x=(x_{ij})$, satisfying all the above requirements will fall into center C, which is now directly checked. Taking the quasi-transvection C as the generating words of the center $t_{ij}(\delta),\ \delta \in AnnR$ $(<i,j>-$ angular positions) and "scalar" words $d_1(\varepsilon_1)\circ \ldots \circ d_n(\varepsilon_n)$, We can formulate the following result here.

Theorem 5. Projective generalized m-triangular group $PET^{o}_{n,m}(R),\ n\ge 2,\ (1\le m\le n)$, over a non-zero associative ring R in generators (Ed) is represented by relations 1–16, angular relations $t_{ij}(\delta)=0,\ \delta\in AnnR\ (i\le n-m,\ j\ge i+m)$, and also "scalar" relations

$$d_1(\varepsilon_1)\circ \ldots \circ d_n(\varepsilon_n)=0$$

$$(\text{here } \varepsilon_k \in centR^{o}).$$

In conclusion, we note that similar questions for monomial groups $Mon_n(R),\ \mathrm{P}Mon_n(R),\ n\ge 2,$ over the associative ring R were previously solved in [6] and [7].

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