ABOUT MINIMAX OPTIMAL CONTROL PROBLEM AN ENSEMBLE OF TRAJECTORIES OF DIFFERENTIAL INCLUSION WITH DELAY

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Abstract. *In this paper we consider one class controlled differential inclusion with a delay argument. For such models of dynamical systems, the minimax optimal control problem for ensemble of trajectories is researched. The existence problem and conditions of optimality is studied. The existence theorem for optimal control, and the necessary and sufficient conditions of optimality are obtained.*

Keywords: *differential inclusion, delay argument, ensemble of trajectories, minimax problem, optimal control, existence, conditions of optimality.*

INTRODUCTION

Differential inclusions are of great interest in the theory of optimal control [1,2,3]. They have important applications in the theory of differential equations with discontinuous right-hand sides, in differential games, in mathematical economics and in other fields. The theory of differential inclusions and their applications is developing in various directions. Research is underway on control and optimization problems for differential inclusions with delays, differential inclusions with a fuzzy right-hand side, and other classes of differential inclusions and their discrete analogues [4–8].

In the theory of optimal control, the study of control systems under conditions of uncertainty is important [9,10]. One of the approaches used when making decisions in conditions of incomplete information about the parameters of the system is the principle of minimizing the guaranteed value of the control quality criterion [11]. This principle leads to minimax (or maximin) control problems, which belong to the class of non-smooth optimization problems [10,12]. A wide class of such problems consists of non-smooth control problems for an ensemble of system trajectories. Various problems of controlling an ensemble of trajectories of a dynamic system are considered in [13–21].

In connection with the issues of controlling dynamic systems with delays under conditions of uncertainty, problems arise of controlling ensembles of differential inclusion trajectories with delays. Some properties of differential inclusions of this type were studied in [22,23]. In particular, conditions for the compactness and convexity of a set of absolutely continuous solutions have been identified, and controllability conditions for an ensemble of trajectories have been studied. For such systems, problems of optimal control of an ensemble of trajectories with non-smooth terminal functional are considered [24,25].

This paper considers the minimax problem of controlling an ensemble of trajectories for one class of model of a dynamic control system with delays. For this problem, existence questions and necessary and sufficient optimality conditions are studied.

2. MATERIALS AND METHODS OF RESEARCH

Let us consider a controlled differential inclusion with delays of the form

$$\frac{dx}{dt} \in A(t)x + \sum_{i=1}^{k} A_i(t)x(t-h_i) + b(t,u), t \in T = [t_0, t_1], \quad (1)$$

where $x \in R^n$, $u \in V$, V is a convex compact set from R^m, $b(t,u)$ is a convex compact set from R^n. We will assume that the right-hand side of differential inclusion (1) satisfies the following conditions:

1) the elements of the matrices $A(t)$ and $A_i(t)$, $i = \overline{1,k}$, are summable on $T = [t_0, t_1]$;

2) a multivalued mapping $(t,u) \to b(t,u)$ is measurable in $t \in T$ and continuous in $u \in V$, and there is a function $\beta(t)$ summable on T such that $\|b(t,u)\| \le \beta(t), \ \forall (t,u) \in T \times V$.

Admissible controls for system (1) are measurable bounded m -vector functions $u = u(t), t \in T$, such that $u(t) \in V$ almost everywhere on T. We denote the set of all admissible controls by $U(T)$.

Let be $u = u(t), t \in T$, an admissible trajectory, $\varphi_0(\cdot) \in C^n(T_0)$, $T_0 = [t_0 - h, t_0]$, where $h = \max_{i=\overline{1,k}} h_i$. An admissible trajectory is each continuous on $T_1 = [t_0 - h, t_1]$ and absolutely continuous on $T = [t_0, t_1]$ a n-vector function $x = x(t)$, satisfying differential inclusion (1) for $u = u(t), t \in T$, and initial condition $x(t) = \varphi_0(t), t \in T_0$. Let us denote $H(u, \varphi_0)$ – the set of all admissible trajectories of system (1), corresponding to the control $u \in U(T)$ and initial function $\varphi_0(\cdot) \in C^n(T_0)$; $X(t,u,\varphi_0) = \{\xi \in R^n : \xi = x(t), x(\cdot) \in H(u,\varphi_0)\}$, $t \in T$ – the corresponding ensemble of trajectories.

Let the state of the ensemble of trajectories of system (1) be estimated by the terminal functional

$$G(u) = \sup_{x(t_1) \in X(t_1,u,\varphi_0)} g(x(t_1)). \quad (2)$$

In (2) we will assume that $g(x)$, $x \in R^n$ is a maximum function of the form

$$g(x) = \max_{i=\overline{1,k}} (P_i z_i, x), \quad (3)$$

where P_i is $r \times n$-matrix, $z_i \in R^r, i = \overline{1,k}$.. Functions of the form (3) are continuous and convex on R^n. The problem of optimal control of an ensemble of trajectories of system (1) consists of minimizing a functional of the form (2), (3):

$$\mathop{Sup}_{\xi \in X(t_1,u,\varphi_0)} g(\xi) \to \min, \quad u \in U(T). \quad (4)$$

Admissible control $u^*(t), t \in T$ satisfying the relation

$$\min_{u \in U(T)} Sup\{g(\xi) : \xi \in X(t_1,u,\varphi_0)\} = Sup\{g(\xi) : \xi \in X(t_1,u^*,\varphi_0)\},$$

will be called optimal control in the minimax problem (4). Let us denote the posed optimal control problem as follows:

$$\left.\begin{aligned}&\sup_{x(\cdot)}\max_{i=\overline{1,k}}(P_i z_i, x(t_1,u))\to\min_{u(\cdot)}\\&\dot{x}\in A(t)x+\sum_{i=1}^{k}A_i(t)x(t-h_i)+b(t,u),\ \mathrm{u}=\mathrm{u(t)}\in \mathrm{V},\ t\in T=[t_0,t_1]\\&x(t)=\varphi_0(t), t\in T_0=[t_0-h,t_0].\end{aligned}\right\}\tag{5}$$

Minimax problem (5) belongs to the class of non-smooth optimal control problems. Minimax optimal control problems for differential inclusions are studied by methods of multivalued and non-smooth analysis [2,10,12]. We will study questions of existence and optimality conditions in the minimax problem (5) .

By virtue of the results of [22–24], the set $X(t_1,u,\varphi_0)$ is a convex compact set R^n and the representation is valid

$$X(t_1,u,\varphi_0)=S(t_1,\varphi_0)+\int_{t_0}^{t_1}F(t_1,t)b(t,u(t))dt,\tag{6}$$

where $F(t,\tau)-n\times n$ is the matrix function satisfying the equation?

$$\frac{\partial F(t,\tau)}{\partial\tau}=-F(t,\tau)A(\tau)-\sum_{i=1}^{k}F(t,\tau+h_i)A_i(\tau+h_i),\ \ \tau\le t,\ \ F(t,t-0)=E,\ F(t,\tau)\equiv 0,\ \ \tau\ge t+0,$$

$$S(t_1,\varphi_0)=F(t_1,t_0)\varphi_0(t_0)+\sum_{i=1}^{k}\int_{t_0}^{t_0+h_i}F(t_1,t)A_i(t)\varphi_0(t-h_i)dt.$$

Using (6), for the support function $X(t_1,u,\varphi_0)$ we obtain the formula:

$$C(X(t_1,u,\varphi_0),\psi)=(S(t_1,\varphi_0),\psi)+\int_{t_0}^{t_1}C(F(t_1,t)b(t,u(t)),\psi)dt\,.\tag{7}$$

Now, applying formula (7), we obtain the relation:

$$\sup_{\xi\in X_1(u,\varphi_0)}g(\xi)=\sup_{\xi\in X_1(u,\varphi_0)}\max_{i=\overline{1,k}}(z_i,P_i\xi)=\max_{i=\overline{1,k}}C(X(t_1,u,\varphi_0),P_i'z_i)=$$

$$=\max_{i=\overline{1,k}}\left[(S(t_1,\varphi_0),P_i'z_i)+\int_{t_0}^{t_1}C(F(t_1,t)b(t,u(t)),P_i'z_i)dt\right].\tag{8}$$

3. RESEARCH RESULTS

First, let us consider the question of the existence of optimal control in problem (5).

Theorem 1. Let, in addition to the above conditions, the support function $C(b(t,u),\psi)$ of the set $b(t,u)$ be convex in $u\in V$. Then in problem (5) there is an optimal control. In the case of strict convexity of the support function $C(b(t,u),\psi)$ in $u\in V$, the optimal control is unique.

Proof. From the formula (8) it easily follows that the functional

$$J(u)=\sup_{\xi\in X(t_1,u,\varphi_0)}q(\xi)$$

is convex on $U(T)$. This functional will be continuous in the metric $L_2^m(T)$ on a convex and closed subset $U(T)\subset L_2^m(T)$. Therefore, due to the results of convex analysis, a solution exists in problem (5). Since problem (5) is another form of writing problem (4), the existence of optimal control can be considered proven. From formula (8) we also obtain strict convexity of the functional

$J(u) = \sup\limits_{\xi \in X(t_1,u,\varphi_0)} q(\xi)$ if the support function $C(b(t,u),\psi)$ is strictly convex in $u \in V$. And this property ensures the uniqueness of optimal control. The theorem has been proven.

Now let us consider the question of necessary and sufficient conditions for optimality in the minimax problem (5).

Let the control $u^*(\cdot) \in U(T)$ be piecewise continuous. We assume that the control $u^*(t)$ is continuous on the left at each discontinuity point. Consider an arbitrary point $\tau \in [t_0, t_1)$ and a sufficiently small number $\varepsilon > 0$ such that $\tau + \varepsilon \in T, v$ *is* an arbitrary vector from V. With these parameters, consider the admissible control

$$u_\varepsilon(t,v) = \begin{cases} u^*(t), & t \notin (\tau-\varepsilon,\tau), \\ v, & t \in (\tau-\varepsilon,\tau), \end{cases} \qquad (9)$$

Using formula (7), we have :

$$C(X(t_1,u_\varepsilon,\varphi_0),\psi) - C(X(t_1,u^*,\varphi_0),\psi) =$$

$$= \int\limits_{\tau-\varepsilon}^{\tau} [C(F(t_1,t)b(t,v)),\psi) - C(F(t_1,t)b(t,u^*(t))),\psi)]dt \ \forall \psi \in R^n. \qquad (10)$$

Consider the function

$$f_i(\varepsilon) = \begin{cases} C(X(t_1,u_\varepsilon,\varphi_0),P_i z_i), \ \varepsilon > 0, \\ C(X(t_1,u^*,\varphi_0),P_i z_i), \ \varepsilon = 0. \end{cases} \qquad (11)$$

Using relation (10), it is easy to verify that for all $i = \overline{1,k}$ function (11) is continuous at point $\varepsilon = 0$. In addition, taking into account formula (10), we have

$$C(X(t_1,u_\varepsilon,\varphi_0),P_i' z_i) - C(X(t_1,u^*,\varphi_0),P_i' z_i) =$$
$$= \varepsilon[C(F(t_1,\tau)b(\tau,v),P_i' z_i) - C(F(t_1,\tau)b(\tau,u^*(\tau)),P_i' z_i)] + o(\varepsilon), \qquad (12)$$

where $\dfrac{o(\varepsilon)}{\varepsilon} \to 0$ at $\varepsilon \to 0$. Let's put:

$$I_\varepsilon = \{i : C(X(t_1,u_\varepsilon,\varphi_0),P_i' z_i) = \max_{i=\overline{1,k}} C(X(t_1,u_\varepsilon,\varphi_0),P_i' z_i)\},$$

$$I_* = \{i : C(X(t_1,u^*,\varphi_0),P_i' z_i) = \max_{i=\overline{1,k}} C(X(t_1,u^*,\varphi_0),P_i' z_i)\}.$$

Using the continuity of function (11) at the point $\varepsilon = 0$ and formula (12), we obtain the following statement.

Lemma 1. There is a number $\varepsilon_0 = \varepsilon_0(\tau,v) > 0$ such that $I_\varepsilon \subset I_*$ for $\forall \varepsilon \in (0,\varepsilon_0)$, i.e. equality is true:

$$\max_{i=\overline{1,k}} C(X(t_1,u_\varepsilon,\varphi_0),P_i' z_i) = \max_{i \in I_*} C(X(t_1,u_\varepsilon,\varphi_0),P_i' z_i) \ \forall \varepsilon \in (0,\varepsilon_0). \qquad (13)$$

Theorem 2. Let $u^*(t)$, $t \in T -$ piecewise continuous optimal control in problem (4). Then for everyone $t \in T$ the equality is true:

$$\min_{v \in V} \max_{i \in I_*} [C(F(t_1,t)b(t,v),P_i' z_i) - C(F(t_1,t)b(t,u^*(t),P_i' z_i)] = 0 \qquad (14)$$

Proof. For definiteness, we will assume that the optimal control $u^*(t)$ is continuous on the left at each point of discontinuity of $(t_0,t_1]$ the function, and continuous on the right at the point

t_0. Let $\tau \in (t_0, t_1], \varepsilon > 0, \tau - \varepsilon > t_0, v \in V$.Consider an arbitrary admissible control of the form (9). Since $u^*(t), t \in T, -$ the control is optimal, it is a solution to problem (5), i.e. taking into account formula (8) we have:

$$\max_{i=\overline{1,k}} C(X(t_1, u, \varphi_0), P_i' z_i) \geq \max_{i=\overline{1,k}} C(X(t_1, u^*, \varphi_0), P_i' z_i) \; \forall u(\cdot) \in U(T).$$

Means,

$$\max_{i=\overline{1,k}} C(X(t_1, u_\varepsilon, \varphi_0), P_i' z_i) \geq \max_{i=\overline{1,k}} C(X(t_1, u^*, \varphi_0), P_i' z_i).$$

Hence, by virtue of (13) we have

$$\max_{i \in I_*} [C(X(t_1, u_\varepsilon, \varphi_0), P_i' z_i) - C(X(t_1, u^*, \varphi_0), P_i' z_i)] \geq 0.$$

Therefore, taking into account (12), we obtain

$$\max_{i \in I_*} [C(F(t_!, \tau) b(\tau, v), P_i' z_i) - C(F(t_!, \tau) b(\tau, u^*(\tau)), P_i' z_i)] + \frac{o(\varepsilon)}{\varepsilon} \geq 0, \forall v \in V.$$

Passing here to the limit at $\varepsilon \to 0$, we obtain

$$\min_{v \in V} \max_{i \in I_*} [C(F(t_1, \tau) b(\tau, v), P_i' z_i) - C(F(t_1, \tau) b(\tau, u^*(\tau), P_i' z_i)] = 0. \quad (15)$$

So, equality (14) is valid for any $t \in (t_0, t_1]$. Passing to the redistribution in (15) at $\tau \to t_0 + 0$ and taking into account the continuity of control $u^*(t)$ on the right at the point t_0, we note that relation (14) is also valid at the point $t = t_0$. The theorem has been proven.

Theorem 3. Let there be an admissible control $u^*(t), t \in T$ and a number $i_* \in I_*$ such that $t \in T$ equality (14) holds for almost all of them. Then $u^*(t)$ is the optimal control in problem (4).
Proof. Let be $u(t), t \in T$ an arbitrary admissible control. Then, by virtue of (8) and (14), we have

$$\begin{aligned}
&\max_{i=\overline{1,k}} C(X(t_1, u, \varphi_0), P_i' z_i) = \max_{i=\overline{1,k}} [(S(t_1, \varphi_0), P_i' z_i) + \int_{t_0}^{t_1} C(F(t_1, t) b(t, u(t)), P_i' z_i) dt] \geq \\
&\geq \max_{i=\overline{1,k}} [(S(t_1, \varphi_0), P_i' z_i) + \int_{t_0}^{t_1} \min_{v \in V} C(F(t_1, t) b(t, v), P_i' z_i) dt] \geq \max_{i \in I_*} [(S(t_1, \varphi_0), P_i' z_i) + \\
&+ \int_{t_0}^{t_1} \min_{v \in V} C(F(t_1, t) b(t, v), P_i' z_i) dt] \geq (S(t_1, \varphi_0), P_{i_*}' z_{i_*}) + \int_{t_0}^{t_1} \min_{v \in V} C(F(t_1, t) b(t, v), P_{i_*}' z_{i*}) dt = \\
&= (S(t_1, \varphi_0), P_{i_*}' z_{i_*}) + \int_{t_0}^{t_1} C(F(t_1, t) b(t, u^*(t)), P_{i_*}' z_{i_*}) dt = \max_{i=\overline{1,k}} C(X_1(u^*, \varphi_0), P_i' z_i)
\end{aligned} \quad (16)$$

According to (8), the last relation will take the form.

$$\max_{\xi \in X(t_1, u, \varphi^0)} g(\xi) \geq \max_{\xi \in X(t_1, u^*, \varphi^0)} g(\xi), \forall u \in U(T).$$

The resulting inequality shows that $u^*(t), t \in T$, is a solution to problem (5), i.e. optimal control in the minimax problem (4). The theorem has been proven.
Example. Let in problem (5) $\quad n = 2, m = 1, k = 3, \; z_1 = 1, z_2 = 2, z_3 = -1$,

$$P_1 = (2, -1), P_2 = (0, -1), P_3 = (1, -1),$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b(t, u) = [-t, t] \times \{u\}, \; V = [-1, 1],$$

$$t_0 = 0, t_1 = 2, \; h = 1, \; \varphi_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; -1 \le t \le 0.$$

According to these data, we obtain the following minimax problem:

$$\sup_{x(\cdot)} \max\{2x_1(2,u) - x_2(2,u), x_2(2,u), x_2(2,u) - x_1(2,u)\} \to \min_{u(\cdot)}$$

$$\dot{x}_1 \in x_2(t-1) + [-t,t], \dot{x}_2 = u(t), |u(t)| \le 1, t \in [0,2],$$

$$x_1(t) = \varphi_{01}(t) = 1, x_2(t) = \varphi_{02}(t) = 0, \; -1 \le t \le 0.$$

For the system under consideration we have :

$$F(2,\tau) = \begin{pmatrix} 1 & 1-\tau \\ 0 & 1 \end{pmatrix}, \; 0 \le \tau \le 1; \; F(2,\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; 1 \le \tau \le 2;$$

$$\sigma_1 = (S(2,\varphi_0), P_1'z_1) + \int_{t_0}^{2} \min_{v \in V} C(F(2,t)b(t,v), P_1'z_1)dt = \frac{9}{2};$$

$$\sigma_2 = (S(2,\varphi_0), P_1'z_1) + \int_{t_0}^{2} \min_{v \in V} C(F(2,t)b(t,v), P_2'z_2)dt = -4;$$

$$\sigma_3 = (S(2,\varphi_0), P_3'z_3) + \int_{t_0}^{2} \min_{v \in V} C(F(2,t)b(t,v), P_3'z_3)dt = -\frac{1}{2};$$

So, $\max_{i=1,2,3} \sigma_i = \sigma_1, i_* = 1$. From the condition of the form (17)

$$\min_{|v| \le 1} C(F(2,t)b(t,v), P_{i1}'z_1) = C(F(t_1,t)b(t,u^*(t)), P_1'z_1) \; \forall t \in [0,2]$$

we define a single function: $u^*(t) = -1, 0 \le t \le \frac{1}{2}$; $u^*(t) = 1, \frac{1}{2} < t \le 2$. For this function, condition (18) is satisfied:

$$\max_{i=1,2,3}[(S(2,\varphi_0), P_i'z_i) + \int_{t_0}^{2} C(F(2,t)b(t,u^*(t)), P_i'z_i)dt] = \max\{\frac{9}{2}; 2; \frac{9}{4}\} = \frac{9}{2}.$$

So, the found function is $u^*(t) = -1, 0 \le t \le \frac{1}{2}$; $u^*(t) = 1, \frac{1}{2} < t \le 2$, is the optimal control in the problem under consideration.

4. DISCUSSION OF RESULTS

Some useful remarks can be made regarding the results obtained.

Remark 1. Let be $I_* = \{i_*\}$ – a singleton set. Then from Theorem 2 it follows that for piecewise continuous optimal control $u^*(t), t \in T$, the equality is true:

$$\min_{v \in V} C(F(t_1,t)b(t,v), P_{i_*}'z_{i_*}) = C(F(t_1,t)b(t,u^*(t)), P_{i_*}'z_{i_*}) \; \forall t \in T. \qquad (17)$$

Remark 2. From relations (16) it follows that if $i_* \in I_*$, i.e.

$$(S(t_1,\varphi_0), P_{i_*}z_{i_*}) + \int_{t_0}^{t_1} C(F(t_1,t)b(t,u^*(t)), P_{i_*}'z_{i_*})dt =$$

$$= \max_{i=\overline{1,k}}[(S(t_1,\varphi_0), P_i'z_i) + \int_{t_0}^{t_1} C(F(t_1,t)b(t,u^*(t)), P_i'z_i)dt], \qquad (18)$$

and condition (17) is satisfied, then the equality is true $\max_{i=\overline{1,k}} \sigma_i = \sigma_{i_*}$, where

$$\sigma_i = (S(t_1, \varphi_0), P_i' z_i) + \int_{t_0}^{t_1} \min_{v \in V} C(F(t_1, t) b(t, v), P_i' z_i) dt .$$

So, according to Theorem 3, in order to find the optimal control, it will be enough to determine such $i_* \in \{1,2,...,k\}$ that $\max_{i=\overline{1,k}} \sigma_i = \sigma_{i_*}$, and after finding the control $u^*(t), t \in T$ from condition (17), check for this number that i_* condition (18) is met. If this is the case, then the constructed control $u^*(t), t \in T$, will be the optimal control in problem (5).

CONCLUSION

The paper considers one minimax-type control problem for differential switching with delays. The problem under study is a problem of optimal control of an ensemble of trajectories of a dynamic system. The existence of a solution to the minimax problem and optimality conditions are studied. Sufficient conditions for the existence of optimal control are given. Theorems on necessary and sufficient conditions for optimality are proven. The results obtained show that they allow the development of an algorithm for constructing optimal control in the considered minimax problem.

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