ON DYNAMICS OF ISING-POTTS MAPPING IN THE FIELD Q_2 AND ITS APPLICATIONS

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Abstract. *In this paper, we study the Ising-Potts in the field of 2-adic numbers the dynamics of its reflection have been studied. This is a reflection all fixed points are found.*

Keywords: *p-adic numbers, Ising-Potts model, fixed points.*

1. Introduction

p-Adic numbers were first described by Kurt Hensel in 1897. The p-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of power series methods into number theory. Their influence now extends far beyond this. For example, the field of p-adic analysis essentially provides an alternative form of calculus.

However, numerous applications of these numbers to theoretical physics have been proposed papers [2],[23] to quantum mechanics and p-adic valued physical observables [7]. A number of p-adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms. New probability models – p-adic probability models were investigated in [3],[6],[8],[11],[12],[13],[14]. In [9] a theory of stochastic processes with values in p-adic and more general non -Archimedean fields was developed, having probability distributions with non-Archimedean values.

In this article, the dynamics of the Ising-Potts reflection for order k =2 in the 2-adic number field is studied. All fixed points of this reflection have been found. We recall that p-adic Ising model (and its generalizations) when p $\neq$ 2 were studied by many authors like as Ganikhodjaev N.N., Rozikov U.A., Mukhamedov F.M.

In [3] it has been constructed p-adic Gibbs measure for Ising model on Z. Moreover, the authors proved a p-adic analogue of Kolmogorov's extention theorem. In [4] it has been shown that there exists a unique p-adic Gibbs measure for Ising model with four competing interaction on specific trees. In [21] it has been described a set of all translation-invariant generalized p-adic Gibbs measures for Ising model on a Cayley tree of order three. In [16] the authors studied dynamics of p-adic Ising-Potts mapping (p $\neq$ 2) and they showed that under some conditions on parameters the Ising-Potts mapping is chaotic. The present paper is a continuation of these works.

The paper is organized as follows. Section 2 presents definitions and known results. In section 3 we study dynamics of 2-adic Ising-Potts mapping.

2. Preliminaries.

p *-adic numbers.*

By Q, as usual, we denote the field of rational numbers. Let p be a fixed prime number, then every $x \neq 0$ can be represented as x = $p^r \frac{n}{m}$, where $r, n \in Z$, m is a positive integer, and n and m are relatively prime with p: (p; n) = 1, (p;m) = 1. The

p-adic norm of x is defined by

$$|x|_p = \begin{cases} p^{-r}, x \neq 0 \\ 0 \ , x = 0 \end{cases} \qquad (2.1)$$

The absolute value $|\cdot|_p$ is non-Archimedean, meaning that it satisfies the strong triangle inequality $|x+y|_p \le max\{|x|_p, |y|_p\}$. We recall a nice property of the norm, i.e. if $|x|_p > |y|_p$ then $|x+y|_p = |x|_p$. Note that this is a crucial property which is proper to the non - Archimedenity of the norm.

Any p -adic number $x \in Q_p$, $x \neq 0$ can be uniquely represented in the form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots), \qquad (2.2)$$

where $\gamma = \gamma(x) \in Z$ and x_j are integers, $0 \le x_j \le p-1$, $x_0 > 0$, $j = 0, 1, 2, \ldots$ In this case $|x|_p = p^{-\gamma(x)}$.

For each $a \in Q_p$, $r > 0$ we denote

$$B(a,r) = \{x \in Q_p : |x-a|_p < r\}$$

and the set of all *p-adic integers*

$$Z_p = \{ x \in Q_p | : |x|_p \le 1 \}.$$

The set $Z_p^* = Z_p \setminus pZ_p$ is called a set of p -adic units.

Theorem 1. [24] The equation $x^2 = a$, $0 \neq a = p^{-\gamma(a)}(a_0 + a_1 p + \cdots)$, $0 \le a_j \le p-1$, $a_0 > 0$ has a solution $x \epsilon Q_p$ iff hold true the following:

i) $\gamma(a)$ is even;

ii) $y^2 \equiv a_0 (mod) p$ is solvable for $p \neq 2$; the equality $a_1 = a_2 = 0$ holds if $p = 2$.

For $a \in Q_p$ and r > 0 we denote

$$B(a,r) = \{x \in Q_p : |x-a|_p < r\}$$

p-adic logarithm is defined by the series

$$log_p(x) = log_p(1 + (x-1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n},$$

which converges for $x \in B_1(1)$; p-adic exponential is defined by

$$exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{\frac{-1}{p-1}})$.

Denote

$$\varepsilon_p = \left\{x \in Q_p : |x-1|_p < p^{\frac{-1}{p-1}}\right\}.$$

This set is the range of the p-adic exponential function. The following fact is well-known.

Lemma 1. [20] The set ε_p has the following properties.

(a) ε_p is a group under multiplication.

(b) If $a, b \in \varepsilon_p$ then the followings are true

$$|a-b|_p < \begin{cases} \frac{1}{2}, p = 2, \\ 1, p \neq 2, \end{cases} \qquad |a+b|_p = \begin{cases} \frac{1}{2}, p = 2, \\ 1, p \neq 2, \end{cases}$$

(d) If $a \in \varepsilon_p$, then there is an element $h \in B(0, p^{\frac{-1}{p-1}})$ such that $a = exp_p(h)$.

Lemma 2. Let $\in B(0, p^{\frac{-1}{p-1}})$. Then

$$|exp_p(x)|_p = 1, \qquad |exp_p(x) - 1|_p = |x|_p, \qquad |log_p(1+x)|_p = |x|_p,$$

$$log_p\left(exp_p(x)\right) = x, \qquad exp_p(log_p(1+x)) = 1 + x.$$

A more detailed description of p-adic calculus and p-adic mathematical physics can be found in [10, 15, 16].

Let (B, X) be a measurable space, where B is an algebra of subsets of X. A function $\mu: B \to Q_p$ is said to be a p-adic measure if for any $A_1, \dots, A_n \in B$ such that $A_i \cap A_j = \emptyset,\ i \neq j,$ the following holds:

$$\mu(\cup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j) .$$

A p-adic measure is called a probability measure if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in B\} < \infty$ (see, [5]).

We call a p-adic measure a probability measure [3] if $\mu(X) = 1$.

3. Dynamics of the Ising-Potts Function in Q_2.

In this section we consider the following Ising-Potts function

$$f_{\theta,k}(x) = \left(\frac{\theta x+1}{x+\theta}\right)^k, \qquad \theta = 1 + 2 + 2^{2n-2}, n \geq 4 . \qquad (3.1)$$

We notice that $f_{\theta,k}(x) = \left(f_{\theta,1}(x)\right)^k$. Moreover, we find all its fixed points and study their behavior.

3.1.The Fixed Points of the Function (3.1)

Let us recall some necessary notions. Let $x^{(0)}$ be a fixed point of an analytic function f and

$$\lambda = \frac{\partial f}{\partial x}(x^{(0)}) .$$

The fixed point $x^{(0)}$ is called *attracting* if $0 < |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$. We denote

$$A(x^{(0)}) = \{x \in Dom(f) : f^n(x) \to x^{(0)}\}$$

the basin of attraction of the attractive fixed point $x^{(0)}$, here $Dom(f)$ stands for the domain of a function f and $f^n = \underbrace{f \circ f \circ \dots \circ f}_{n}$. [25]

We are going to find all fixed points of the function (3.1). Note that $Dom(f_\theta x) = Q_p \setminus \{-\theta\}$.

In what follows, for the sake of simplicity, we assume that $p = 2$, $k = 2$, $0 < |\theta - 1|_2 = \frac{1}{2}$. Namely, the fixed points are $x_0 = 1$ and

$$x_1 = \frac{(\theta-1)^2 - 2 + (\theta-1)\sqrt{(\theta-1)^2 - 4}}{2} \qquad (3.2)$$

$$x_2 = \frac{(\theta-1)^2 - 2 - (\theta-1)\sqrt{(\theta-1)^2 - 4}}{2} \qquad (3.3)$$

Theorem 3.1. $x_0,\ x_1, x_2$ fixed points will be *indifferent fixed points.*

Proof. Let x_i be a fixed point of (3.1). Then we have

$$f'_{\theta,2}(x_i) = \frac{2(\theta x_i + 1)(\theta^2 - 1)}{(x_i + \theta)^3}, \qquad (3.4)$$

(i) From (3.4) we get

$$f'_{\theta,2}(x_0) = \frac{2(\theta - 1)}{\theta + 1}$$

Since $|\theta - 1|_2 = \frac{1}{2}$ and $|\theta + 1|_2 = \frac{1}{4}$ using non-Archimedean norm's property we obtain

$$\left|f'_{\theta,2}(x_0)\right|_2 = \left|\frac{2(\theta-1)}{\theta+1}\right|_2 = 1.$$

which means x_0 that is an indifferent fixed point.

(ii) Using (3.2) and (3.3) one can calculate that

$$\left|\theta x_{1,2}+1\right|_2=\frac{1}{4},\quad \left|x_{1,2}+\theta\right|_2=\frac{1}{4},\ |\theta-1|_2=\frac{1}{2}\ ,\quad |\theta+1|_2=\frac{1}{4}.$$

Putting these into (3.4) we can easily get

$$\left|f_{\theta,2}'(x_{1,2})\right|_2=\left|\frac{2(\theta x_{1,2}+1)(\theta^2-1)}{(x_{1,2}+\theta)^3}\right|_2=1.$$

So that $x_0,\ x_1$ and x_2 are indifferent fixed points.

Corollary 3.1. *Let* $r=|2^n(\theta-1)|_2$. *Then one has* $B_r(x_1)\cap B_r(x_2)\neq\emptyset$.

Proof. It is enough to show that $x_1\in B_r(x_2)$. From (3.2) and (3.3) , using non-Archimedean norm's property we have

$$|x_1-x_2|_2=\left|(\theta-1)\sqrt{(\theta-1)^2-4}\right|_2\leq r$$

which yields $x_1\in B_r(x_2)$.

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