NILPOTENTS ZERO DIVISORS OF A MULTIDIMENSIONAL MATRIX

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Abstract. *In this article, the necessary conditions for nilpotency of matrices of three and higher dimensions are studied. In addition, its application to quadratic stochastic operators is presented.*

Keywords: *nilpotent, quadratic stochastic operators, zero divisor, multidimensional matrix, simplex.*

НИЛЬПОТЕНТЫ ДЕЛИТЕЛИ НУЛЯ МНОГОМЕРНЫХ МАТРИЦЫ

Аннотация. *В данной статье изучаются необходимые условия нильпотентной матриц трех и более размерностей. Кроме того, представлено его приложение к квадратичным стохастическим операторам.*

Ключевые слова: *нильпотент, квадратическое стохастическое операторы, делители нуля, многомерная матрица, симплекс.*

INTRODUCTION

We know that quadratic stochastic operators depend on $P_{ij,k}$ coefficients, that is way we need to study three-dimensional matrices. Although matrices have been widely studied in $n=2$, but the properties of matrices in three and higher dimensions have not been fully studied. This article is devoted to it and shows its connection with quadratic stochastic operators.

MATERIALS AND METHODS

We introduce the necessary concepts of multidimensional matrices and quadratic stochastic operators. We mainly need three-dimensional matrices, so we give their definitions below.

Definition. Let $T_1,\ T_2,\ T_3,...,\ T_m$ be $k\times n$ matrices with real entries. A 3-d matrix A is an array of the matrices $T_1,\ T_2,\ T_3,...,\ T_m$ arrayed from top to bottom. We denote this as $A=[T_1|T_2|...|T_m]_T$ and call this the top representation of A. We also write A as follows. **[1]**

$$A=\left[\left[\begin{matrix} a_{1,1,1} & a_{1,1,2} & \cdots & a_{1,1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,k,1} & a_{1,k,2} & \cdots & a_{1,k,n} \end{matrix}\right]\middle|\cdots\middle|\left[\begin{matrix} a_{m,1,1} & a_{m,1,2} & \cdots & a_{m,1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,k,1} & a_{m,k,2} & \cdots & a_{m,k,n} \end{matrix}\right]\right]_T \quad (1)$$

A is a $m\times k\times n$ matrix.

Definition. The nonlinear stochastic operators are defined on a simplex S, and the dimensional of the simplex is $(m-1)$, as **[3]**

$$S^{m-1}=\{x_i=(x_1,x_2,x_3,...,x_n)\in R^m,\sum_{i=1}^{m}x_i=1,\text{ and } x_i\geq 0\}$$

Definition. The evaluation of the nonlinear stochastic operators as

$$(Vx)_k=\sum_{i,j=1}^{m}P_{ij,k}x_ix_j$$

Where the $P_{ij,k}$ is the transaction matrix under the condition:

$$P_{ij,k} = P_{ji,k} \geq 0, \sum_{k=1}^{m} P_{ij,k} = 1.$$

We write the operator in the form of a matrix. []

$$\sum_{i=1}^{m} x_i p_{ij,k} x_j = \begin{pmatrix} (x_1 \quad x_2 \quad \dots \quad x_m) \begin{pmatrix} a_{11,1} & \cdots & a_{1m,1} \\ \vdots & \ddots & \vdots \\ a_{m1,1} & \cdots & a_{mm,1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \\ (x_1 \quad x_2 \quad \dots \quad x_m) \begin{pmatrix} a_{11,2} & \cdots & a_{1m,2} \\ \vdots & \ddots & \vdots \\ a_{m1,2} & \cdots & a_{mm,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \\ \vdots \qquad \vdots \qquad \vdots \\ (x_1 \quad x_2 \quad \dots \quad x_m) \begin{pmatrix} a_{11,m} & \cdots & a_{1m,m} \\ \vdots & \ddots & \vdots \\ a_{m1,m} & \cdots & a_{mm,m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \end{pmatrix} \quad (2)$$

If we multiply and calculate this matrix, we can write each in the following form.

$$V(x_i) = \begin{pmatrix} V(x_1) = a_{11,1}x_1x_1 + a_{12,1}x_1x_2 + \dots + a_{1m,1}x_1x_m + a_{21,1}x_2x_1 + a_{22,1}x_2x_2 \\ + \dots + a_{2m,1}x_2x_m + \dots + a_{m1,1}x_mx_1 + a_{m2,1}x_mx_2 + \dots + a_{mm,1}x_mx_m \\ V(x_2) = a_{11,2}x_1x_1 + a_{22,2}x_1x_2 + \dots + a_{1m,2}x_1x_m + a_{21,2}x_2x_1 + a_{22,2}x_2x_2 \\ + \dots + a_{2m,2}x_2x_m + \dots + a_{m1,2}x_mx_1 + a_{m2,2}x_mx_2 + \dots + a_{mm,2}x_mx_m \\ \vdots = \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots + \vdots \\ V(x_m) = a_{11,m}x_1x_1 + a_{12,m}x_1x_2 + \dots + a_{1m,m}x_1x_m + a_{21,m}x_2x_1 + a_{22,m}x_2x_2 \\ + \dots + a_{2m,m}x_2x_m + \dots + a_{m1,m}x_mx_1 + a_{m2,m}x_mx_2 + \dots + a_{mm,m}x_mx_m \end{pmatrix} \quad (3)$$

In our work, it is clear that a nonlinear operator can be constructed using multidimensional matrices.

RESULTS

1. For multidimensional matrices **[2]**, we will consider which conditions are fulfilled for divisors and nilpotency of zero, and we will construct a nonlinear operator using those matrices. Of course, for zero to have divisors, the following condition must be fulfilled $A \cdot B = 0$. We calculate from the property of multidimensional multiplication.

$$A = \left[\begin{pmatrix} a_5 & a_8 \\ a_6 & a_7 \end{pmatrix} \middle| \begin{pmatrix} a_1 & a_4 \\ a_2 & a_3 \end{pmatrix} \right]_T \quad B = \left[\begin{pmatrix} b_5 & b_8 \\ b_1 & b_4 \end{pmatrix} \middle| \begin{pmatrix} b_6 & b_7 \\ b_2 & b_3 \end{pmatrix} \right]_F$$

$$T_1^1=\begin{pmatrix}a\\c\end{pmatrix}\quad T_2^1=\begin{pmatrix}m\\p\end{pmatrix}\quad F_1^1=\begin{pmatrix}a\\m\end{pmatrix}\quad F_2^1=\begin{pmatrix}c\\p\end{pmatrix}$$

$$T_1^2=\begin{pmatrix}b\\d\end{pmatrix}\quad T_2^2=\begin{pmatrix}n\\q\end{pmatrix}\quad F_1^2=\begin{pmatrix}b\\n\end{pmatrix}\quad F_1^1=\begin{pmatrix}d\\q\end{pmatrix}$$

$$\overline{T}_1=\begin{bmatrix}T_1^1\cdot F_1^1 & T_1^2\cdot F_1^2\\ T_1^1\cdot F_2^1 & T_1^2\cdot F_2^2\end{bmatrix}=\begin{bmatrix}a_5b_5+a_6b_1 & a_8b_8+a_7b_4\\ a_5b_6+a_6b_2 & a_8b_7+a_7b_3\end{bmatrix}$$

$$\overline{T}_2=\begin{bmatrix}T_2^1\cdot F_1^1 & T_2^2\cdot F_1^2\\ T_2^1\cdot F_2^1 & T_2^2\cdot F_2^2\end{bmatrix}=\begin{bmatrix}a_1b_5+a_2b_1 & a_4b_8+a_3b_4\\ a_1b_6+a_2b_2 & a_4b_7+a_3b_3\end{bmatrix}$$

$$A\cdot B=\left[\begin{pmatrix}a_5b_5+a_6b_1 & a_8b_8+a_7b_4\\ a_5b_6+a_6b_2 & a_8b_7+a_7b_3\end{pmatrix}\middle|\begin{pmatrix}a_1b_5+a_2b_1 & a_4b_8+a_3b_4\\ a_1b_6+a_2b_2 & a_4b_7+a_3b_3\end{pmatrix}\right]_T$$

$$a_5b_5+a_6b_1=0,\quad a_8b_8+a_7b_4=0,\quad a_1b_5+a_2b_1=0,\quad a_4b_8+a_3b_4=0$$
$$a_5b_6+a_6b_2=0,\quad a_8b_7+a_7b_3=0,\quad a_1b_6+a_2b_2=0,\quad a_4b_7+a_3b_3=0$$

When we solve this system of equations, the possible solutions are as follows.

$$0,0,0,0,0,0,0,0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8$$
$$0,0,c,d,0,0,a_7,a_8,b_1,b_2,0,0,b_5,b_6,0,0$$
$$a_1,a_2,0,0,a_5,a_6,0,0,0,0,b_3,b_4,0,0,b_7,b_8$$
$$a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,0,0,0,0,0,0,0,0$$

2. In our next problem, we will consider nilpotency. $n=2, A^2=0$

$$A=\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix}\middle|\begin{pmatrix}m & n\\ p & q\end{pmatrix}\right]_T\quad A=\left[\begin{pmatrix}a & b\\ m & n\end{pmatrix}\middle|\begin{pmatrix}c & d\\ p & q\end{pmatrix}\right]_F$$

$$T_1^1=\begin{pmatrix}a\\c\end{pmatrix}\quad T_2^1=\begin{pmatrix}m\\p\end{pmatrix}\quad F_1^1=\begin{pmatrix}a\\m\end{pmatrix}\quad F_2^1=\begin{pmatrix}c\\p\end{pmatrix}$$

$$T_1^2=\begin{pmatrix}b\\d\end{pmatrix}\quad T_2^2=\begin{pmatrix}n\\q\end{pmatrix}\quad F_1^2=\begin{pmatrix}b\\n\end{pmatrix}\quad F_1^1=\begin{pmatrix}d\\q\end{pmatrix}$$

$$\overline{T}_1=\begin{bmatrix}T_1^1\cdot F_1^1 & T_1^2\cdot F_1^2\\ T_1^1\cdot F_2^1 & T_1^2\cdot F_2^2\end{bmatrix}=\begin{bmatrix}a^2+mc & b^2+dn\\ ac+pc & bd+dq\end{bmatrix}$$

$$\overline{T}_2=\begin{bmatrix}T_2^1\cdot F_1^1 & T_2^2\cdot F_1^2\\ T_2^1\cdot F_2^1 & T_2^2\cdot F_2^2\end{bmatrix}=\begin{bmatrix}ma+mp & nb+qn\\ mc+p^2 & nd+q^2\end{bmatrix}$$

$$A^2=\left[\begin{pmatrix}a^2+mc & b^2+dn\\ ac+pc & bd+dq\end{pmatrix}\middle|\begin{pmatrix}ma+mp & nb+qn\\ mc+p^2 & nd+q^2\end{pmatrix}\right]_T$$

$$a^2+mc=0,\ ma+mp=0$$
$$ac+pc=0,\ mc+p^2=0$$
$$b^2+dn=0,\ nb+qn=0$$
$$bd+dq=0,\ nd+q^2=0$$

3. The solution can be in the following forms.

$$(0,0,0,0,0,0,0,0),\ (0,0,0,0,0,n,0,0),\ (0,0,0,0,m,0,0,0),\ (0,0,0,0,m,n,0,0),$$
$$(0,b,0,-\frac{b^2}{n},0,n,0,-b),\ (0,b,0,-\frac{b^2}{n},m,n,0,-b),\ (a,0,-\frac{a^2}{m},0,m,0,-a,0),$$
$$(a,0,-\frac{a^2}{m},0,m,n,-a,0),\ (a,b,-\frac{a^2}{m},-\frac{b^2}{n},m,n,-a,-b).$$

This is our problem, when does the multiplication of the matrix by itself become equal to itself again $A^2=A$?

$$A=\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\middle|\begin{pmatrix} m & n \\ p & q\end{pmatrix}\right]_T \quad A=\left[\begin{pmatrix} a & b \\ m & n\end{pmatrix}\middle|\begin{pmatrix} c & d \\ p & q\end{pmatrix}\right]_F$$

$$A^2=\left[\begin{pmatrix} a^2+mc & b^2+dn \\ ac+pc & bd+dq\end{pmatrix}\middle|\begin{pmatrix} ma+mp & nb+qn \\ mc+p^2 & nd+q^2\end{pmatrix}\right]_T$$

$$a^2+mc=a,\ ma+mp=m$$
$$ac+pc=c,\ mc+p^2=p$$
$$b^2+dn=b,\ nb+qn=n$$
$$bd+dq=d,\ nd+q^2=q$$

$(a,b,c,d,m,n,p,q)=\{(1,0,0,0,0,0,0,0),(1,0,0,0,0,0,0,1),$
$(1,0,0,0,0,0,1,0),(1,0,0,0,0,0,1,1),(1,0,0,0,0,n,0,1),(1,0,0,0,0,n,1,1),$
$(1,0,0,0,m,0,0,0),(1,0,0,0,m,0,0,1),(1,0,0,0,m,n,0,1),$
$(1,1,0,0,0,0,0,0),(1,1,0,0,0,0,0,1),(1,1,0,0,0,0,1,0),$
$(1,1,0,0,0,0,1,1),(1,1,0,0,0,n,0,0),(1,1,0,0,0,n,1,0),$
$(1,1,0,0,m,0,0,0),(1,1,0,0,m,0,0,1),(0,0,0,0,0,0,0,0),$
$(0,0,0,0,0,0,0,1),(0,0,0,0,0,0,1,0),(0,0,0,0,0,0,1,1),(0,0,0,0,0,n,0,1),$
$(0,0,0,0,0,n,1,1),(0,0,0,0,m,0,1,0),(0,0,0,0,m,0,1,1),(0,0,0,0,m,n,1,1),$
$(0,1,0,0,0,0,0,0),\ (0,1,0,0,0,n,1,1)(0,1,0,0,m,0,1,0),(0,1,0,0,m,0,1,0),$
$(0,1,0,0,0,0,1,0),(0,1,0,0,0,0,1,1),(0,1,0,0,0,n,0,0),(0,1,0,0,0,n,0,0),(0,1,0,0,0,n,1,0),$
$(0,1,0,0,m,0,1,1),(0,1,0,0,m,n,1,0),(0,b,0,\frac{b(1-b)}{n},0,n,0,1-b),$,
$(0,b,0,\frac{b(1-b)}{n},m,n,1,1-b),(1,1,0,0,m,n,0,0),(1,b,0,\frac{b(1-b)}{n},0,n,0,1-b),$

$$(1,b,0,\frac{b(1-b)}{n},0,n,1,1-b),(1,b,0,\frac{b(1-b)}{n},m,n,0,1-b),(a,0,\frac{a(1-a)}{m},0,m,0,1-a,0),$$
$$(a,0,\frac{a(1-a)}{m},0,m,0,1-a,1),(a,0,\frac{a(1-a)}{m},0,m,n,1-a,1),(a,1,\frac{a(1-a)}{m},0,m,0,1-a,0),$$
$$(0,b,0,\frac{b(1-b)}{n},0,n,1,1-b),(a,1,\frac{a(1-a)}{m},0,m,0,1-a,1),(a,1,\frac{a(1-a)}{m},0,m,n,1-a,0)$$
$$(a,b,\frac{a(1-a)}{m},\frac{b(1-b)}{n},m,n,1-a,1-b).$$

DISCUSSION

We have shown that there are many solutions for nilpotency and zero divisors. Through these for each matrix, a nonlinear stochastic operator **[4]-[14]** can be construct. Example: If coefficient (1,1,0,0,0,0,1,1) for a three-dimensional matrix, from (3) we see its expression on the operator.
$$\begin{matrix} V(x_1)=x_1^2+x_1x_2 \\ V(x_2)=x_2^2+x_1x_2 \end{matrix}.$$

CONCLUSIONS

In short, we want calculate multiplication of matrix, zero divisors, nilpotency and construction of an operator using three-dimensional matrices.

REFERENCES

1. Don. D. Vo. 3-dimensional matrix mathematical operations. Dissertation of master's science, California state University, pp. 60, 2017
2. N. P. Sokolov, Spartial matrices and their applications (Russian), Gosudarstv. Izdat. Fiz. -Mat. Lit., Moscow, 1960, pp,300 MR24-A122.
3. R. Abdulghafor. S. Almotairi. H. Almohamedh. B. Almutairi. A. Bajahzar. S.S. Almutairi. A nonlinear convergence consensus: Extreme doubly stochastic quadratic operators for multi-agent systems. MDPI. Basel. Symmetry. Pp.19. 2020.
4. R.A.Brualdi and J.Csima,"On the plane term rank of a three dimensional matrix,"Proceedings of the American Mathematical Society,vol.54,no.1,pp.471–473,197
5. F.Shahidi, R. Ganikhodzaev and R. Abdulghafor. The dynamics of some extreme doubly stochastic quadratic operators. Middle east journal of scientific research.Malaysia. January 2013.
6. B.C. Arnold. Majorization: here, there and everywhere. Statistical scince. institute of mathematical statistics, Vol.22, No. 3, 2007. 407-413
7. R. Turkmena, E. Vehbi. B. Paksoy , F. Zhang. Some inequalities of majorization type. Linear algebra and its applications 437 (2012) 1305-1316.
8. Majorization, Doubly Stochastic Matrices, and Comparison of Eigenvalues T. Ando Division of Applied Mathematics Research Institute of Applied Electricity Hokkaido University, Sapporo, Japan
9. Constantin P. Niculescu and Lars-Erik Persson convex functions a contemporary approach second expanded edition May 23, 2018.
10. J.W.Moon. Topics on tournament. University of Alberta. 2013

11. U. A. Rozikov. Evolution Operator and Algebras of Sex-linked inheritance. Asia Pacific Mathematics Newslette. Volume 3. Number 1. January 2013
12. R.N.Ganikhodzhaev, F. Mukhammedov. U. A. Rozikov. Quadratic stochastic operators and processes results and open problems. Infinite Dimensional Analysis, Quantum Probability and Related Topics. Vol. 14, No. 2. 20011. 279-335.
13. A.W. Marshall, I. Olkin, B. C. Arnold, Inequalities: Theory of majorization and its application, springer, New York, 2011.
14. R. Kaneiwa, N-dimensional matrices and nth order invariant forms (Japanese). Characterization of arithmetic functions (Proc. Sympos., Res. Inst. Math. Sci., Kyoto University, Kyoto, (1975). Surikaisekiken yusho Kokyuroku 274(1976), 83-97. MR 58-10955.