ABOUT THE TIME OPTIMAL CONTROL PROBLEM FOR AN ENSEMBLE OF TRAJECTORIES OF DIFFERENTIAL INCLUSION WITH DELAY

Otakulov Salim
Doctor of Physical and Mathematical Sciences, Professor, Jizzakh Polytechnic Institute
Kholiyarova Feruza Khafizovna
Senior Lecturer, Samarkand Branch of the Tashkent University of Information Technologies named after Muhammad Al -Khwarizmi

https://doi.org/10.5281/zenodo.7041752

Abstract. *The paper considers the time optimal control problem for an ensemble of trajectories of one class of differential inclusions with delay. In this problem, necessary and sufficient optimality conditions are obtained. The application of these results to a linear time optimal control problem with a terminal set in the form of a moving half-space is shown. For this case, using the obtained optimality conditions the main stages of solving the problem is indicated.*

Keywords: *mathematical model, control system, differential inclusion, ensemble of trajectories, time optimal control problem, optimality conditions.*

О ЗАДАЧЕ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ПО БЫСТРОДЕЙСТВИЮ ДЛЯ АНСАМБЛЯ ТРАЕКТОРИЙ ДИФФЕРЕНЦИАЛЬНОГО ВКЛЮЧЕНИЯ С ЗАПАЗДЫВАНИЕМ

Аннотация. *В работе рассматривается задача управления по быстродействию для ансамбля траекторий одного класса дифференциальных включений с запаздыванием. В этой задаче получены необходимые и достаточные условия оптимальности. Показано применение этих результатов к линейной задаче быстродействия с терминальным множеством вида подвижного полупространства. Для этого случая с помощью полученных условий оптимальности указаны основные этапы решения задачи.*

Ключевые слова: *математическая модель, система управления, дифференциальное включение, ансамбль траекторий, задача быстродействия, условия оптимальности.*

INTRODUCTION

The mathematical theory of optimal processes is of great practical importance for actual control problems from various fields of economics, technology and production [1]. The practical application of optimization methods contributes to achieving the goal of efficient use of natural and energy resources and determining the optimal parameters in various engineering and technical developments and projects.

In optimal control theory, dynamic models described by differential inclusions are of great interest as models of continuous dynamical systems [2,3,4]. Differential inclusions, which originally appeared as generalized differential equations, are also used in the theory of differential equations with discontinuous right-hand sides, in problems of mechanics, in differential games, in mathematical economics, and in other areas. In the theory of differential inclusions, research is being carried out in various directions [4–12].

Differential inclusions with control parameters are of great interest for studying models of control systems in informational constraints. Mathematical models of control problems, taking into account the incompleteness of information about the parameters of external influences and initial data, measurement inaccuracies in the process of observation, and other uncertainties, constitute an important class in the theory of dynamic control systems [8–11]. For such models, called control systems under uncertainty, control problems arise for an ensemble of trajectories [8, 9]. Various problems related to the control of an ensemble of trajectories are considered in [10–20].

Differential inclusions with delays depending on the control parameter are of great interest. Some properties of the set of solutions of such a class of differential inclusions and their application to the control problem for an ensemble of trajectories are considered in [6, 7, 13,14]. Therefore, the development of research on control problems for an ensemble of trajectories for differential inclusions with a retarded argument is of great theoretical and practical importance. In this paper, developing the studies [14,15], we study the problem of optimal control of an ensemble of trajectories for one class of differential inclusions with a delay argument.

MATERIALS AND METHODS

Consider a dynamic control system described by a differential inclusion with a delay of the form

$$\dot{x} \in A(t)x + A_1(t)x(t-h) + b(t,u), \ \ t \geq t_0 \, (1)$$

where $\dot{x} = \frac{dx}{dt}$, $x-$is the n-state vector, $u-$ is the m- control vector, $b(t,u) \subset R^n$, R^n is the Euclidean space of n-vectors with scalar product $(x,y) = \sum_{i=1}^{n} x_i y_i$ and norm $\|x\| = \sqrt{(x,x)}$.

We will study the control system (1) under the following assumptions:

a) the elements $n \times n$-matrices $A(t)$ and $A_1(t)$ are integrable on any finite interval $[t_0, t_1]$;

b) for $\forall (t,u) \in T \times V$ a set $b(t,u)$, a convex compact set from R^n;

c) for any $t_1 > t_0$ multi-valued mapping is $(t,u) \to b(t,u)$ measurable with respect to $t \in T = [t_0, t_1]$ and continuous with respect to $u \in V$, and there exists an integrable function $\beta(t), t \in T$ such that

$$\|b(t,u)\| = \sup_{\gamma \in b(t,u)} \|\gamma\| \leq \beta(t), \ \ \forall (t,u) \in T \times V.$$

Definition 1. As an admissible control we will consider every measurable bounded on some interval $[t_0, t_1]$ m-vector function $u = u(t)$, taking almost everywhere on $[t_0, t_1]$ values from a convex compact $V \subset R^m$.

Denote by U_T - the set of all admissible controls $u = u(t)$ defined on the interval $T = [t_0, t_1]$.

Definition 2. An admissible trajectory corresponding to a control $u(\cdot) \in U_T$ and a continuous initial function $\varphi_0(t), t \in T_0 = [t_0 - h, t_0]$ is called continuous on $T_1 = [t_0 - h, t_1]$ and absolutely continuous on $T = [t_0, t_1]$ n-vector function $x = x(t)$ satisfying almost everywhere on the T differential inclusion (1) and the initial condition

$$x(t) = \varphi_0(t), \ \ t \in T_0. \quad (2)$$

Denote by the $H_T(u, \varphi_0)$ set of all admissible trajectories of system (1) corresponding to the control $u(\cdot) \in U_T$ and satisfying the initial condition (2).

In control problems for systems described by differential inclusions, the concept of an ensemble of trajectories is important. This concept is defined through the sets

$$X_T(t,u,\varphi_0)=\{\xi = x(t): x(\cdot)\in H_T(u,\varphi_0)\},\ \ t\in T=[t_0,t_1].$$

Geometric representation of the dependence of a given set on time $t\in T$, i.e. multi-valued mapping $t\to X(t,u,\varphi_0)$, $t\in T$, we will call the ensemble of trajectories of system (1) corresponding to the control $u(\cdot)\in U_T$ and the initial function $\varphi_0(\cdot)$.

Assume that we are given a convex, closed, and bounded set $M(t)\subset R^n$, that continuously depends on $t\ge t_0$.

Statement of the optimal control problem. Let us consider the problem of transferring the ensemble of trajectories of system (1), (2) in the shortest time to a position that ensures the possibility for the trajectories of the system to reach a given moving terminal set, i.e. need *to find control* $u^*(t),\ t\in[t_0,t_1^*]$, *which minimizes the time* $t_1=t_1(u)$, *at which the terminal constraint is valid*

$$X_T(t_1,u,\varphi_0)\cap M(t_1)\ne\varnothing,\ \exists u(\cdot)\in U_T. \qquad (3)$$

Definition 3. The control $u^*(t),\ t\in[t_0,t_1^*]$, for which $t_1(u^*)$ is the minimum execution time of the terminal constraint (3), we call *the optimal control* , and $t_1^*=t_1(u^*)$ is *the optimal time.*

The main goal of the work is to find out the conditions from which it will be possible to find the optimal time and optimal control.

From the statement of the problem and the purpose of the study, it is clear that we have to use the properties of the set of absolutely continuous solutions and the corresponding ensemble of trajectories of the differential inclusion (1).

It follows from the results of [6,7] that for each $u(\cdot)\in U_T$ and every continuous initial function the $\varphi_0(\cdot)$ set $H_T(u,\varphi_0)$ is a convex compact set from the space of continuous on $T_1=[t_0-h,t_1]$ n-vector of functions, and $X_T(t,u,\varphi_0)$ is a convex compact set from R^n. Moreover, multi-valued mappings $u\to H_T(u,\varphi_0)$ and $(t,u)\to X_T(t,u,\varphi_0)$ are continuous. According to the results of [13, 14], the representation

$$X_T(t,u,\varphi_0)=S(t,\varphi_0)+\int_{t_0}^{t}F(t,\tau)b(\tau,u(\tau))d\tau, \qquad (4)$$

where $F(t,\tau)$ is $n\times n$-matrix function satisfying the equation

$$\frac{\partial F(t,\tau)}{\partial\tau}=-F(t,\tau)A(\tau)-F(t,\tau+h)A_1(\tau+h),\ \ \tau\le t,$$

$$F(t,t-0)=E,\ \ F(t,\tau)\equiv 0,\ \ \tau\ge t+0,$$

$$S(t,\varphi_0)=F(t,t_0)\varphi_0(t_0)+\int_{t_0}^{t_0+h}F(t,\tau)A_1(\tau)\varphi_0(\tau-h)d\tau,$$

E is the identity $n\times n$ matrix.

Let's put: $\Phi(t)=\bigcup_{u\in U_T}X_T(t,u,\varphi_0)-M(t),\ \ t\in T=[t_0,t_1]$. Using representation (4) and the properties of the integral of multivalued mappings, it is easy to show that the set $\Phi(t)$ satisfies the equality

$$\Phi(t) = S(t,\varphi_0) + \int_{t_0}^{t} F(t,\tau)b(\tau,V)dx - M(t), \qquad (5)$$

where

$$b(t,V) = \bigcup_{v\in V} b(t,v).$$

By virtue of formula (5) and the properties of the integral of multi-valued mappings [3,4], the set $\Phi(t)$ is a convex compact set R^n and continuously depends on $t \in [t_0, t_1]$.

We introduce the following function

$$\mu(t,\psi) = \sup_{u\in U_T} C(X_T(t,u,\varphi_0),\psi) + C(-M(t),\psi).\ t \in T = [t_0,t_1].\psi \in R^n. \qquad (6)$$

Using (4), we obtain the formula for the support function

$$C(X_T(t,u,\varphi_0),\psi) = \sup_{\xi\in X(t,u,\varphi_0)} (\xi,\psi)$$

of the set $X_T(t,u,\varphi_0)$:

$$C(X_T(t,u,\varphi_0),\psi) = (S(t,\varphi_0),\psi) + \int_{t_0}^{t} C(F(t,\tau)b(\tau,u(\tau)),\psi)d\tau.$$

From the results of [14], it easily follows that the function $\mu(t,\psi)$ defined by formula (6) is continuous on $[t_0,t_1]\times R^n$ for all $t_1 > t_0$, and the equality

$$\mu(t,\psi) = (S(t,\varphi_0),\psi) + \int_{t_0}^{t} \max_{v\in V} C(F(t,\tau)b(\tau,v),\psi)d\tau + C(-M(t),\psi). \qquad (7)$$

From continuity function $\mu(t,\psi)$ and representation (7) it follows that the function

$$\gamma(t) = \inf_{\|\psi\|=1}\left[(S(t,\varphi_0),\psi) + \int_{t_0}^{t} \max_{v\in V} C\big(F(t,\tau)b(\tau,v),\psi\big)d\tau + C\big(-M(t),\psi\big)\right] \qquad (8)$$

is continuous at $t \geq t_0$. These presented results form the basis of the method for studying the stated speed problem.

RESULTS

1. Optimality conditions. Let us study the necessary and sufficient optimality conditions in the given time-optimal problem.

It is clear that relation (3) holds if and only if $0 \in X_T(t_1,u,\varphi_0) - M(t_1)$, . $\exists u(\cdot) \in U_T$ Hence, the terminal constraint (3) is equivalent to the condition $0 \in \Phi(t_1)$. Thus, for the optimal moment of time t_1^*, the formula is valid:

$$t_1^* = \inf\{t_1 : 0 \in \Phi(t_1)\}. \qquad (9)$$

Considering that for a convex compact set $\Phi(t_1)$ the relation is $0 \in \Phi(t_1)$ equivalent to the inequality $\inf_{\|\psi\|=1} C(\Phi(t_1),\psi) \geq 0$, and the function $\mu(t_1,\psi)$ defined by formula (6) is the support function of the set $\Phi(t_1)$, from equality (9) we obtain

$$t_1^* = \inf\left\{t_1 : \inf_{\|\psi\|=1} \mu(t_1,\psi) \geq 0\right\}. \qquad (10)$$

By virtue of formulas (7) and (8), we have $\gamma(t) = \inf_{\|\psi\|=1} \mu(t,\psi)$. Using the continuity of the function $\gamma(t)$, $t \geq t_0$, from (10) we obtain

$$t_1^* = \inf\left\{t_1 : \inf_{\|\psi\|=1} \mu(t_1,\psi) \geq 0\right\} = \min\left\{t_1 : \inf_{\|\psi\|=1} \mu(t,\psi) = 0\right\}.$$

So the following is true

Theorem 1. The optimal time t_1^* in the time-optimal problem is the minimum root of the equation

$$\gamma(t) = 0,\ t \geq t_0, \qquad (11)$$

where the function $\gamma(t)$ has the form (8).

Now, using Theorem 1 and using the results of [14], we present the necessary and sufficient optimality conditions in the time-optimal problem.

Theorem 2. Let $u^*(t)$, $t \in [t_0, t_1^*]$ is the optimal control and t_1^* is the optimal time. Then the equality

$$C(F(t_1^*,t)b(t,u^*(t)),\psi^*) = \max_{v\in V} C(F(t_1^*,t)b(t,v),\psi^*), \qquad (12)$$

holds for almost all $t \in [t_0, t_1^*]$, where vector $\psi^* \in R^n$, $\|\psi^*\| = 1$, such that

$$\mu(t_1^*,\psi^*) = \min_{\|\psi\|=1} \mu(t_1^*,\psi), \qquad (13)$$

the function is $\mu(t,\psi)$ defined by formula (8).

Theorem 3. Let be t_1^* the minimum root of equation (11), and the vector $\psi^* \in R^n$, $\|\psi^*\| = 1$, satisfies condition (13). Suppose that a measurable bounded control $u^*(t)$ for almost all $t \in T^* = [t_0, t_1^*]$ satisfies equality (12), and from this equality the value $u^*(t)$ is uniquely determined. Then $u^*(t)$, $t \in T^* = [t_0, t_1^*]$, is the optimal control in the time-optimal problem.

2. Application to a linear time-optimal problem with a terminal set of the form of a moving half-space. General scheme (stages) of the algorithm. Let system (1) be linear in control, i.e., $b(t,u) = B(t)u + Q(t)$, $u \in V \subset R^m$ where $B(t)$ is a $n \times m$ matrix whose elements are integrable on each finite interval $T = [t_0, t_1]$, the multi-valued mapping is $t \to Q(t)$ measurable on T, and $\|\xi\| \leq q_0(t)$, $\forall \xi \in Q(t)$, $q_0(\cdot) \in L_1(T)$, strongly convex compact $V \subset R^m$. As a terminal set, consider a moving half-space $M(t) = \{\xi \in R^n : (\xi, a) \leq \sigma(t)\}$, where $\sigma(t)$ is a continuous function at $t \geq t_0$. According to the results obtained, in this case, we can propose the following general scheme of the algorithm - the main stages of solving the speed problem.

1. Calculate the matrix $F(t,\tau)$ and find $S(t,\varphi_0)$ for $t \geq t_0$.
2. Find a feature

$$\gamma(t) = (S(t,\varphi_0),-a) + \int_{t_0}^{t} \max_{v\in V}(F(t,\tau)B(\tau)v,-a)d\tau + \int_{t_0}^{t} C(F(t,\tau)Q(\tau),-a)d\tau + \sigma(t). \qquad (14)$$

3. Determine the minimum root t_1^* of the equation $\gamma(t)=0, t \geq t_0$.
4. Determine the function $u^*(t)$, $t \in [t_0, t_1^*]$ satisfying the equality

$$(F(t_1^*,t)B(t)u^*(t),a) = \min_{v \in V}(F(t_1^*,t)B(t)v,a). \qquad (15)$$

Then, t_1^* - optimal time, $u^*(t)$ - optimal control.

DISCUSSION

In the paper, for a differential inclusion of the form (1), the time optimal control problem is considered. This problem differs from problems for deterministic control systems in that it is required to control not a separate, isolated trajectory, but their totality - an ensemble of trajectories. Moreover, the mobility of the terminal set is allowed, i.e. time dependence: $M = M(t), t \geq t_0$. Movable terminal sets are often encountered in applied problems. In particular, as such a set, one can take the set of possible phase states of another dynamical system controlled independently of the control system under consideration. The formulation of the problem is also interesting for control systems in game situations.

The results obtained give necessary and sufficient conditions for optimality in the studied time-optimal problem. Theorem 1 indicates that by solving an equation of the form (11), one can determine the optimal time. The necessary optimality conditions are given in Theorem 2 in the form of relations (12) and (13). Theorem 3 shows that the requirement to uniquely determine the control $u^*(t)$, $t \in [t_0, t_1^*]$ from condition (12) is sufficient for the optimality of this control. This condition will be satisfied, for example, if the support function is $C(b(t,u),\psi)$ strictly concave in $u \in V$ for all $t \geq t_0$ and $\psi \in R^n$.

CONCLUSIONS

According to the results obtained, the construction of an optimal control in the considered time-optimal problem can be carried out in the following steps: first, find the minimum root t_1^* of equation (11); then we find the vector $\psi^* \in R^n$, $\|\psi^*\| = 1$, satisfying condition (13); after that, the optimal control $u^* = u^*(t), t \in T$, is determined from the maximum condition (12). The solution method based on this scheme is presented for a linear control system (1) with a moving terminal set in the form of a half-space. This result can be developed for terminal sets of other kinds. The proposed method constitutes a theoretical basis for the development of computational algorithms for solving the speed problem of the considered type.

REFERENCES

1. Afanasiev V.N., Kolmanovsky V.V., Nosov V.V. Mathematical theory of designing control systems. – Moscov: Visshaya shkola, 2003.
2. Blagodatskikh V.I., Filippov A.F. Differential inclusions and optimal control. Proceedings of the Mathematical Institute of the Academy of Sciences of the USSR, V.169, 1985. - p. 194-252.
3. Borisovich Yu.G., Gelman B.D., Myshkis A.D., Obukhovsky V.V. Introduction to the theory of multivalued mappings and differential inclusions. -M.: Komkniga., 2005.
4. Polovinkin E.S. Multivalued analysis and differential inclusions. –M.: Fizmatlit , 2015.

5. Minchenko L.I., Tarakanov A.N. Methods of multivalued analysis in the study of control problems for differential inclusions with delay. Reports of BSUIR, No. 1, 2004. - p. 27-37.
6. Otakulov S. On the minimization problem of reachable set estimation of control system. IFAK Workshop on Generalized Solutions in Control Problems (GSCP–2004). Pereslavl-Zalessky , Russia, September 22-26, 2004. –p. 212-217.
7. Otakulov S., Kholiyarova F.Kh. On the theory of controlled differential inclusions with retarded argument. Reports of the Academy of Sciences of the Republic of Uzbekistan ., No. 3, 2005. - p. 14-17.
8. Clark F. Optimization and non-smooth analysis. – M.: Nauka, 1988.
9. Kurzhansky A.B. Management and supervision in conditions of uncertainty. –M.: Nauka, 1977.
10. Kurzhanskaya N.A. Optimal control of a beam of trajectories of a system with underdetermined parameters. Differential Equations, Vol. 21, No. 3, 1985. - p. 404-410.
11. Ovsyannikov D.A. Mathematical methods for controlling bundles of trajectories. Leningrad: Publishing House of Leningrad State University, 1981.
12. Otakulov S. Problems of control of an ensemble of trajectories of differential inclusion. Lambert academic Publishing , 2019.
13. Otakulov S. , Kholiyarova F.Kh. Time optimal control problem of ensemble trajectories of differential inclusion with delays. Journal of Advanced Research in Dynamical and Control Systems, vol.12, issue 6, 2020. -p. 1043-1050.
14. Otakulov S., Kholiyarova F.Kh. The problem of speed control of an ensemble of trajectories of a differential inclusion with delay. Academic Research in Educational Sciences, 2021, vol.2 , issue 3. - p . 778-788.
15. Otakulov S., Kholiyarova F.Kh. On The Problem of Controllability an Ensemble of Trajectories for One Information Model of Dynamic Systems with Delay. International Conference on Information Science and Communications Technologies (ICISCT-2020). Tashkent, November 4-6, 2020. Publiser : IEEE . - p.1-4 .
16. Otakulov S., Rahimov B. Sh. Haydarov T.T. On the property of relative controllability for the model of dynamic system with mobile terminal set. AIP Conference Proceedings, 2022, 2432, 030062. -p. 1–5.
17. Otakulov S., Rahimov B. Sh. On the structural properties of the reachability set of a differential inclusion. Proceedings of International Conference on Research Innovations in Multidisciplinary Sciences, March 2021. New York, USA. -p. 150-153.
18. Otakulov S., Rahimov B. Sh. About the property of controllability an ensamble of trajectories of differential inclusion. International Enjineering Journal for Research & Development. Vol.5, issue 4, 2020. pp.366-374.
19. Otakulov S.,Haydarov T.T. The nonsmooth control problem for dinamic system with parameter under conditions of incomplete initial date. International Conference On Innovation Perspectives, Psychology and Social Studiees(ICIPPCS-2020), may 11-12 2020. International Enjineering Journal for Research & Development(IEJRD). pp.211-214.