KUBIK TENGLAMANI YECHISHNING KARDANO USULI

Kulanov Ikrom Burxonovich

Jizzax politexnika instituti katta o'qituvchisi

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Annotatsiya. *Bu maqolada kubik tenglamani yechishning Kardano usuli keltirilgan. Bu usul bilan kubik tenglamalarni yechish iqtidorli o'quvchilarning matematika faniga bo'lgan qiziqishlarini ortirishga, mustaqil ravishda bilim saviyalarini oshirishga va mantiqiy fikrlashga yordam beradi.*

Kalit so'zlar: *bessel funksiyasi, kubik tenglama,Gardano, qoshma kompleks sonlar.*

МЕТОД КАРДАНО ДЛЯ РЕШЕНИЯ КУБИЧЕСКОГО УРАВНЕНИЯ

Аннотация. *В данной статье представлен метод Кардано решения кубического уравнения. Решение кубических уравнений таким способом помогает одаренным учащимся повысить интерес к математике, самостоятельно повысить уровень знаний, логически мыслить.*

Ключевые слова: *функции Басселя, кубическое уравнение, Кардано, сопрежённые комплексное числа.*

CARDANO METHOD FOR SOLVING A CUBIC EQUATION

Abstract. *This article presents Cardano's method for solving a cubic equation. Solving cubic equations in this way helps gifted students increase their interest in mathematics, independently increase their level of knowledge, and think logically.*

Keywords: *bessel function, sequations of the third degree, Cardano, complex conjugate.*

Kirish. Tenglamalar nazariyasi Bessel funksiyalarining[1] nollari, integrasiya va boshqalar xarakteristik tenglamalarni o'rganishda zarur bo'lgan tenglamalarni yechishni o'z ichiga oladi.

Ko'phad yoki integral ratsional algebraik funksiya

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

ko'rinishda bo'ladi, bu yerda $a_0, a_1, a_2, \dots, a_n$ o'zgarmas koeffitsentlar, n esa ko'phadning darajasi va nomanfiy butun son.

$n = 1,2,3,4$ bo'lganda mos chiziqli, kvadrat, kubik va bikvadrat funksiyalar deyiladi. O'zgarmasni nol darajali ko'phad sifatida qarash mumkin.

Algebraik funksiya - $P_0(\mathrm{x})y^n + P_1(\mathrm{x})y^{n-1} + P_2(\mathrm{x})y^{n-2} + \cdots + P_n(\mathrm{x}) = 0$ ko'rinishdagi tenglamani qanoatlantiruvchi xar qanday $y = f(x)$ funksiyadir. Bu yerda $P_0(\mathrm{x}), P_1(\mathrm{x}), P_2(\mathrm{x}), \dots, P_n(\mathrm{x})$ lar x o'zgaruvchining ko'phadlari.

Algebraik tenglamalarda yechimlarini tenglamaning ildizlari (yoki nollari) sifatida ham qarash mumkin.chim

$ax + b = 0$ chiziqli tenglama uchun yechim $x = -\frac{b}{a}$, kvadrat tenglama uchun yechim $x_{1,2} = \frac{-b\pm\sqrt{D}}{2a}$, kubik tenglamaning yechimlarini esa Kardano usulida topish mumkin.

Kubik tenglamani XI asrda Umar Xayyom (1048-1123) birinchi marta geometrik usulda yechgan edi. U uchinchi darajali tenglamani aylana va parabola tenglamalariga ajratib ularning kesishish nuqtasining berilgan tenglamaning yechimi ekanligini isbotlagan edi[2].

Italyan matematigi Djerolamo Kardano kubik tenglamani yechishning bu usulini 1545 yilda Ars Magna shahrida e'lon qilgan.

Asosiy qism. Kubik tenglamaning umumiy ko'rinishi

$$a_1x^3 + b_1x^2 + c_1x + d_1 = 0 \quad (1)$$

[1] B.V.Ramana: "Higher Engineering Mathematics" 11th Edition, Tata McGraw-Hill, 2010.34 p.
[2] G.Gaymnazarov va boshq. Umar Xayyom va algebra //Fizika, matematika va informatika. 2014.№ 5.– B.48-52.

Tenglamaning ikkala tomonini a_1 ga bo'lib

$$x^3 + bx^2 + cx + d = 0 \ (2)$$

ko'rinishga olib kelamiz.

Bu tenglamani $x = u - \frac{b}{3}$ (3) almashtirish orqali soddaroq holga keltiramiz:

$$(u - \frac{b}{3})^3 + b(u - \frac{b}{3})^2 + c(u - \frac{b}{3}) + d = 0$$

$$\left(u^3 - 3u^2\frac{b}{3} + 3u\frac{b^2}{9} - \frac{b^3}{27}\right) + b\left(u^2 - 2u\frac{b}{3} + \frac{b^2}{9}\right) + c\left(u - \frac{b}{3}\right) + d = 0 \text{ yoki}$$

$$u^3 + pu + q = 0 \ (4)$$

Bu yerda $p = c - \frac{b^2}{3}, q = d - \frac{bc}{3} + \frac{2b^3}{27}$. (5)

Bu kubik tenglamani yechish uchun $u = y + z$ (6) almashtirish olamiz.

$$(y + z)^3 = u^3$$

$$y^3 + 3y^2z + 3yz^2 + z^3 = u^3$$

$$y^3 + z^3 + 3yz(y + z) = u^3$$

$$u^3 - 3yzu - y^3 - z^3 = 0 \ (7)$$

4 va 7 solishtirib $p = -3yz, q = -(y^3 + z^3)$ ni olamiz. Bu yerdan

$$yz = -\frac{p}{3} \text{ yoki } y^3z^3 = -\frac{p^3}{27}.$$

Demak, $= -(y^3 + z^3), y^3z^3 = -\frac{p^3}{27}$. Viyet teoremasiga ko'ra x^3, y^3 lar $t^2 + qx - \frac{p^3}{27} = 0$ (8) kvadrat tenglamaning yechimi bo'ladi. Shuning uchun

$$y^3 = \frac{-q + \sqrt{q^2 + 4q\frac{p^3}{27}}}{2} = \frac{-q}{2} + \sqrt{\frac{q^2}{4} + q\frac{p^3}{27}} \ (9) \text{ va}$$

$$z^3 = \frac{-q - \sqrt{q^2 + 4q\frac{p^3}{27}}}{2} = \frac{-q}{2} - \sqrt{\frac{q^2}{4} + q\frac{p^3}{27}} \ (10)$$

$\Delta = \frac{q^2}{4} + q\frac{p^3}{27}$ belgilab olamiz[3].

1-holat. Faraz qilaylik, $\Delta > 0$ bo'lsa, u holda y^3 va z^3 ikkalasi ham haqiqiy va (4) kubik tenglamaning ildizlari $y+z, \omega y + \omega^2 z, \omega^2 y + \omega z$ (12) ga teng bo'ladi. Bu yerda

$\omega = \frac{-1+\sqrt{3}i}{2}$, $\omega^2 = \frac{-1-\sqrt{3}i}{2}$ va $\omega \cdot \omega^2 = \omega^3 = 1$ $(1, \omega, \omega^2$ ildizlar shundayki $1 + \omega + \omega^2 = 0)$.

Demak (2) kubik tenglamaning izlanayotgan ildizlari quyidagiga teng bo'ladi.

$$\begin{cases} x_1 = u_1 - \frac{b}{3} = y + z - \frac{b}{3} \\ x_2 = u_2 - \frac{b}{3} = \omega y + \omega^2 z - \frac{b}{3} \\ x_3 = u_3 - \frac{b}{3} = \omega^2 y + \omega z - \frac{b}{3} \end{cases} \quad (13)$$

2-holat. Faraz qilaylik, $\Delta = 0$ bo'lsa, u holda y^3 va z^3 ikkalasi ham haqiqiy va (4) kubik tenglamaning ildizlari $y+z, \omega y + \omega^2 z, \omega^2 y + \omega z$ yoki $2y, -y, -y$ (14) ga teng bo'lad($\omega + \omega^2 = -1, y = z$).

Demak (2) kubik tenglamaning izlanayotgan ildizlari quyidagiga teng bo'ladi.

$$\begin{cases} x_1 = 2y - \frac{b}{3} \\ x_2 = -y - \frac{b}{3} \\ x_3 = -y - \frac{b}{3} \end{cases} \quad (15)$$

3-holat. $\Delta < 0$ bo'lsa, u holda y^3 va z^3 ikkalasi ham komples sonlar va $y^3 = a + ib, z^3 = a - ib$ ga teng bo'ladi.

Agar kubik tenglamaning yechimlari $m + in$ *va* $m - in$ bo'lsa u holda (4) kubik tenglamaning yechimlari quyidagicha bo'ladi.

$$\begin{cases} x_1 = y + z = m + in + m - in = 2m \\ x_2 = \omega y + \omega^2 z = \omega(m+in) + \omega^2(m-in) = -m - \sqrt{3}n \\ x_3 = \omega^2 y + \omega z = \omega^2(m+in) + \omega(m-in) = -m + \sqrt{3}n \end{cases} \quad (16)$$

3 Roman Wituła, Damian Słota. Cardano's formula, square roots, Chebyshev polynomials and radicals. Journal of Mathematical Analysis and Applications. 363 (2010) 639–647p.

Muavr formulasidan foydalanib bu yechimlarni quyidagicha ifodalash mumkin.

(4) kubik tenglamaning yechimlari quyidagicha bo'lsin.

$$u = y + z = (a+ib)^{1/3} + (a-ib)^{1/3} \quad (17)$$

$a = r\cos\varphi, b = r\sin\varphi. (r = \sqrt{a^2+b^2}, tg\varphi = \frac{b}{a}$. larni o'rniga qo'yamiz

$$(a+ib)^{1/3} = (r(\cos\varphi + i\sin\varphi))^{1/3} = r^{1/3}\left(\cos\frac{\varphi+2\pi k}{3} + i\sin\frac{\varphi+2\pi k}{3}\right), k = 0,1,2. \quad (18)$$

Xuddi shunday

$$(a-ib)^{1/3} = (r(\cos\varphi - i\sin\varphi))^{1/3} = r^{1/3}\left(\cos\frac{\varphi+2\pi k}{3} - i\sin\frac{\varphi+2\pi k}{3}\right), k = 0,1,2. \quad (19)$$

Shunday qilib (18), (19) larni (17) ga qo'yib

$$u = r^{1/3}\left(\cos\frac{\varphi+2\pi k}{3} + i\sin\frac{\varphi+2\pi k}{3}\right) + r^{1/3}\left(\cos\frac{\varphi+2\pi k}{3} - i\sin\frac{\varphi+2\pi k}{3}\right) = 2r^{1/3}\cos\frac{\varphi+2\pi k}{3}.$$

$$k = 0\ da\ 2r^{1/3}\cos\frac{\varphi}{3}, k = 1\ da\ 2r^{1/3}\cos\frac{\varphi+2\pi}{3}, k = 2\ da\ 2r^{1/3}\cos\frac{\varphi+4\pi}{3}$$

Natijada (2) kubik tenglamaning yechimi

$$\begin{cases} x_1 = u_1 - \frac{b}{3} = 2r^{1/3}\cos\frac{\varphi}{3} - \frac{b}{3} \\ x_2 = u_2 - \frac{b}{3} = 2r^{1/3}\cos\frac{\varphi+2\pi}{3} - \frac{b}{3} \\ x_3 = u_3 - \frac{b}{3} = 2r^{1/3}\cos\frac{\varphi+4\pi}{3} - \frac{b}{3} \end{cases} \quad (20)$$

(13),(15),(20) sonlar kubik tennglama uchun Kardano formulalari deyiladi[4].

Misol 1. $28x^3 - 9x^2 + 1 = 0$ tenglamani yeching[5].

Yechilishi.Tenglamani $x^3 - \frac{9x^2}{28} + \frac{1}{28} = 0$ ko'rinishga keltiramiz.

$$x = u - \frac{b}{3} = u - \frac{1}{3}\left(-\frac{9}{28}\right) = u + \frac{3}{28}$$

O'rniga qoyamiz $\left(u + \frac{3}{28}\right)^3 - \frac{9\left(u+\frac{3}{28}\right)^2}{28} + \frac{1}{28} = 0$ yoki $u^3 - \frac{27}{28^2}u + \frac{730}{28^3} = 0$

Demak, $p = -\frac{27}{28^2}, q = \frac{730}{28^3}$ $u = y + z$ deb olamiz.

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[4] B.V.Ramana: "Higher Engineering Mathematics" 11th Edition, Tata McGraw-Hill, 2010.35p.

[5] Бронштейн И.Н., Семендяев К.А. Справочник по математике для инженеров и учащихся ВТУЗов. М.: Наука, 2006.

Shuday qilib $y^3+z^3=-q=-\frac{730}{28^3}$ va $y^3z^3=-\frac{p^3}{27}=-\frac{27^3}{28^6}\cdot\frac{1}{27}=-\frac{27^2}{28^6}$

y^3, z^3 lar t boyicha quyidagi kvadrat tenglamaning yechimlari bo'ladi.

$$t^2+\frac{730}{28^3}t-\frac{27^2}{28^6}=0$$

Bu yerda diskirminant

$$D=\left(\frac{730}{28^3}\right)^2+4\frac{27^2}{28^6}>0$$

Shuning uchun x va y larning qiymatlari xaqiqiy sonlardan iborat. t uchun yechim

$$t_{1,2}=\frac{\left(-\frac{730}{28^3}\pm\sqrt{\left(\frac{730}{28^3}\right)^2+4\frac{27^2}{28^6}}\right)}{2}=\frac{-730\pm 728}{2\cdot 28^3}$$

Shuning uchun

$$y^3=\frac{-730+728}{2\cdot 28^3}=-\frac{2}{28^3}, yoki\ y=-\frac{1}{28}, z^3=\frac{-730-728}{2\cdot 28^3}=-\frac{729}{28^3}=-\left(\frac{9}{28}\right)^3, yoki\ z=-\frac{9}{28}$$

Shunday qilib, $=y+z=-\frac{1}{28}-\frac{9}{28}=-\frac{10}{28}$, Demak

$x=u+\frac{3}{28}=-\frac{10}{28}+\frac{3}{28}=-\frac{7}{28}=-\frac{1}{4}$ Bu berilgan kubik tenglamaning yechimi, qolgan ikkita yechimi

$$\omega y+\omega^2 z-\frac{b}{3}=\left(\frac{-1+\sqrt{3}i}{2}\right)\left(-\frac{1}{28}\right)+\left(\frac{-1-\sqrt{3}i}{2}\right)\left(-\frac{9}{28}\right)=\frac{10+8\sqrt{3}i}{2\cdot 28}=\frac{5+4\sqrt{3}i}{28}$$

Demak, $x=u+\frac{3}{28}=\frac{5+4\sqrt{3}i}{28}+\frac{3}{28}=\frac{8+4\sqrt{3}i}{28}=\frac{2+\sqrt{3}i}{7}$.

$$\omega y+\omega^2 z-\frac{b}{3}=\left(\frac{-1-\sqrt{3}i}{2}\right)\left(-\frac{1}{28}\right)+\left(\frac{-1+\sqrt{3}i}{2}\right)\left(-\frac{9}{28}\right)=\frac{10-8\sqrt{3}i}{2\cdot 28}=\frac{5-4\sqrt{3}i}{28}$$

Demak, $x=u+\frac{3}{28}=\frac{5-4\sqrt{3}i}{28}+\frac{3}{28}=\frac{8-4\sqrt{3}i}{28}=\frac{2-\sqrt{3}i}{7}$.

Shunday qilib kubik tenglamaning uchala ildizi quyidagiga teng:

$-\frac{1}{4}, \frac{2+\sqrt{3}i}{7}, \frac{2-\sqrt{3}i}{7}$.

Misol 2. $x^3 - 27x + 54 = 0$ tenglamani yeching[6].

Yechilishi. Bu yerda $p = -27, q = 54. x^2$ yoqligi uchun almashtirish olmaymiz.

$x = y + z$ deb olamiz.

Shuday qilib $y^3 + z^3 = -q = -54$ va $y^3 z^3 = -\frac{p^3}{27} = -\frac{(-27)^3}{27} = 27^2$

y^3, z^3 lar t boyicha quyidagi kvadrat tenglamaning yechimlari bo'ladi.

$t^2 + 54t - 27^2 = 0$

Bu yerda diskirminant $D = (54)^2 + 4 \cdot 27^2 = 0$

Demak, t ikkita bir xil yechimga ega.Ya'ni $x^3 = y^3, x = y$.

Demak, $y^3 = \frac{-54}{2} = -27$ yoki $y = -3 = z$

Berilgan kubik tenglamaning yechimlari: $2y, y, -y$ ya'ni 6,-3,-3.

Misol 3.$x^3 - 3x^2 - 12x + 16 = 0$ tenglamani yeching.

x^2 ni yoqotish uchun $x = u - \frac{b}{3} = u - \frac{(-3)}{3} = u + 1$ almashtirish olamiz.

U holda $(u + 1)^3 - 3(u + 1)^2 - 12(u + 1) + 16 = 0$

$u^3 + 3u^2 + 3u - 3u^2 - 6u - 3 - 12u - 12 + 16 = 0$ yoki

$u^3 - 15u + 2 = 0$

Bu yerda $p = -15, q = 2. u = y + z$ almashtirish olamiz.Shunday qilib

$y^3 + z^3 = -q = -2$ va $y^3 \cdot z^3 = -\frac{p^3}{27} = -\frac{(-15)^3}{27} = 5^3$.

Shunday qilib y^3, z^3lar t bo'yicha quyidagi kvadratik tenglamaning yechimi bo'ladi.

$t^2 + 2t + 5^3 = 0$

Bu yerda diskirminant $D = (2)^2 - 4 \cdot 5^3 < 0$

Shunday qilib y^3, z^3 ildizlar qo'shma kompleks sonlar.

$$y^3 = \frac{-2 + \sqrt{4 - 4 \cdot 125}}{2} = -1 + \sqrt{124}i$$

[6] A. S. Yunusov, S. I. Afonina, M. A. Berdiqulov,D. I. Yunusova qiziqarli matematikava olimpiada masalalari.55b.

Demak,

$a=-1, b=\sqrt{124}, r=\sqrt{a^2+b^2}=\sqrt{1+124}=\sqrt{125}, tg\varphi=\frac{b}{a}=\frac{\sqrt{124}}{-1}=-\sqrt{124}$

, demak,

$y=\left(-1+\sqrt{124}i\right)^{1/3}=r^{1/3}\left(cos\frac{\varphi+2\pi k}{2}+isin\frac{\varphi+2\pi k}{2}\right)=\sqrt{5}\left(cos\frac{\varphi+2\pi k}{2}+isin\frac{\varphi+2\pi k}{2}\right). k=0,1,2.$

Xuddi shunday

$$z^3=\frac{-2-\sqrt{4-4\cdot 125}}{2}=-1-\sqrt{124}i$$

Demak,

$a=-1, b=-\sqrt{124}, r=\sqrt{a^2+b^2}=\sqrt{1+124}=\sqrt{125}, tg\varphi=\frac{b}{a}=\frac{-\sqrt{124}}{-1}=\sqrt{124}$

, demak,

$z=\left(-1-\sqrt{124}i\right)^{1/3}=r^{1/3}\left(cos\frac{\varphi+2\pi k}{2}-isin\frac{\varphi+2\pi k}{2}\right)=\sqrt{5}\left(cos\frac{\varphi+2\pi k}{2}-isin\frac{\varphi+2\pi k}{2}\right). k=0,1,2.$

$u=y+z, x=u+1$ bo'lgani uchun berilgan kubik tenglamaning yechimlari:

$1+2\sqrt{5}cos\frac{\varphi}{3}, 1+2\sqrt{5}cos\frac{\varphi+2\pi}{3}, 1+2\sqrt{5}cos\frac{\varphi+4\pi}{3}$, bu yerda $\varphi=arctg(-\sqrt{124})$.

Xulosa. Turli matematika musobaqalari va olimpiadalarida kubik tenglamalarga bir necha bor duch kelamiz. Bu tenglamalarni yechishda o'quvchilar ancha qiyinchiliklarga duch kelishadi. Yuqorida keltirilgan kubik tenglamanin yechishning Kardano usuli bu qiyinchiliklarni yengishga yordam beradi.To'rtinchi darajali tenglamalarni yechish usullaririni keying maqolada e'lon qilamiz.

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