AN EXAMPLE OF A REGULAR SPACE ON WHICH EVERY CONTINUOUS FUNCTION IS CONSTANT

A.A. Zaitov

Professor, Doktor of physics and mathematics, Teacher at Tashkent Institute of Engineers and Construction

D. R. Atamuradova

Teacher at Tashkent State Pedagogical University

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Abstract: *This article gives examples of a regular space where each continuous function is constant.*

Key words: *regular space, continuous function, constant*

ПРИМЕР РЕГУЛЯРНОГО ПРОСТРАНСТВА, НА КОТОРОМ ЛЮБАЯ НЕПРЕРЫВНАЯ ФУНКЦИЯ ПОСТОЯННА

Аннотация: *В этой статье приведены примеры регулярного пространства, где каждая непрерывная функция постоянна.*

Ключевые слова: *регулярное пространство, непрерывная функция, константа.*

Let X be an arbitrary infinite set of cardinality $\mathfrak{m} \geq \aleph_0$, $x_0 \in X$ be some point. Let's define a family

$$\tau = \{U \subset X : x_0 \notin U\} \cup \{U : |X \setminus U| < \infty\}.$$

It is easy to establish that (X, τ) is a topological space that is denoted by $A(\mathfrak{m})$. For every $x \in X \setminus \{x_0\}$ the single-point set $\{x\}$ is clopen, and the set $\{x_0\}$ is closed, but not open. At the same time, U is an open neighborhood of x_0 if and only if U contains x_0, and it has a finite complement. A family consisting of all single-point sets $\{x\}$, $x \in X \setminus \{x_0\}$, and of all the sets U with a finite complement, form a base of the space(X, τ).

Closure of a subset $A \subset A(\mathfrak{m}) = (X, \tau)$ (i. e. the set of all points $x \in X$, each open neighborhood of which intersects with A) in this case, *is determined* by equality

$$\bar{A} = \begin{cases} A, & \text{if A is finite,} \\ A \cup \{x_0\}, & \text{if A is infinite.} \end{cases}$$

Indeed, if A is finite and $x \notin A$ and $x \neq x_0$ the set $\{x\}$ is an open neighborhood of x and $\{x\} \cap A = \emptyset$; if $x_0 \notin A$, that set $U = X \setminus A$ is an open neighborhood of x_0 и $U \cap A = \emptyset$. Thus, if A is finite, then every point of X, not included in A, has a neighborhood that does not intersect with A. So, $\bar{A} = A$.

If A is infinite and $x \notin A$, then in the case $x \neq x_0$ the set $\{x\}$ is an open neighborhood of x and $\{x\} \cap A = \emptyset$. This means that $x \notin \bar{A}$. Each neighborhood of a point x_0 has the form $U = X \setminus F$, where F is finite set. Since A cannot be embedded into any finite set F by any way, then the sets U and A necessarily intersect. Hence, by definition of the closure, we have

$$\bar{A} = A \cup \{x_0\}.$$

It follows that:

(Cl_1) an infinite subset of the space $A(\mathfrak{m}) = (X, \tau)$ is closed if and only if it contains x_0;

(Cl_2) an intersection of any two closed infinite subsets of the space $A(\mathfrak{m}) = (X, \tau)$ is nonempty.

The point x of the topological space X is called an accumulation point (a limit point) of the set $A \subset X$, if $x \in \overline{A \setminus \{x\}}$, that is, each neighborhood Ox of a point x has at least one point y other than x belonging to the intersection:

$$y \in A \cap Ox.$$

Note that the point x_0 is a unique accumulation point of the topological space $A(\mathfrak{m}) = (X, \tau)$.

For a topological space $A(\mathfrak{m}) = (X, \tau)$ *interior* $IntA$ of a subset $A \subset X$ (i. e. the set of all points $x \in A$, each of which has an open neighborhood lying in A) is defined by the equality

$$IntA = \begin{cases} A, & \text{if } X \setminus A \text{ is finite,} \\ A \setminus \{x_0\}, & \text{if } X \setminus A \text{ is infinite.} \end{cases}$$

Indeed, if $X \setminus A$ is finite, then for each $x \in A$, $x \neq x_0$, its open neighborhood $\{x\}$ contained in A. If $x_0 \in A$, then $U = X \setminus (X \setminus A) = A$ is an open neighborhood of x_0 and $U \subset A$. So, in this case, $IntA = A$.

Let now $X \setminus A$ be infinite. Then for each $x \in A$, $x \neq x_0$, its open neighborhood $\{x\}$ contains in A. But, each neighborhood $U = X \setminus F$ of x_0, where F is a finite set, does not contain in A. Therefore $x_0 \notin IntA$, i. e. $IntA = A \setminus \{x_0\}$.

It follows that:

(Int_1) any two open subsets of the space $A(\mathfrak{m}) = (X, \tau)$ with a finite complement have a nonempty intersection;

(Int_2) a subset of the space $A(\mathfrak{m}) = (X, \tau)$ with infinite complement is open if and only if it does not contain x_0.

Constancy sets of continuous functions on spaces of the type $A(\mathfrak{m})$

For an arbitrary continuous function $\varphi: X \to \mathbb{R}$ and for every $i \in \mathbb{N}$ a set $\varphi^{-1}\left(\left(\varphi(x_0) - \frac{1}{i},\ \varphi(x_0) + \frac{1}{i}\right)\right)$ is an open neighborhood of a point x_0 (as a preimage of an open set $\left(\varphi(x_0) - \frac{1}{i},\ \varphi(x_0) + \frac{1}{i}\right)$). Then the property (Int_2) implies that the set $X_i = X \setminus \varphi^{-1}\left(\left(\varphi(x_0) - \frac{1}{i},\ \varphi(x_0) + \frac{1}{i}\right)\right)$ cannot be infinite. Hence every X_i is finite, $i = 1, 2, \ldots$. Therefore $X_0 = \bigcup_{i \in \mathbb{N}} X_i$ is no more than countably. Also, since $x_0 \notin X_i$ for every $i = 1, 2, \ldots$, than $x_0 \notin X_0$. For a point

$$x \in X \setminus X_0 = X \setminus \bigcup_{i \in \mathbb{N}} X_i = \bigcap_{i \in \mathbb{N}} (X \setminus X_i) =$$

$$= \bigcap_{i \in \mathbb{N}} \left(\varphi^{-1} \left(\left(\varphi(x_0) - \frac{1}{i}, \varphi(x_0) + \frac{1}{i} \right) \right) \right) =$$

$$= \varphi^{-1} \left(\bigcap_{i \in \mathbb{N}} \left(\varphi(x_0) - \frac{1}{i}, \varphi(x_0) + \frac{1}{i} \right) \right)$$

we have $\varphi(x) \in \bigcap_{i \in \mathbb{N}} \left(\varphi(x_0) - \frac{1}{i}, \varphi(x_0) + \frac{1}{i} \right)$. That is why $\varphi(x) = \varphi(x_0)$ for all $x \in X \setminus X_0$.

Thus, we establish the following properties. (1_0) For an arbitrary continuous function $\varphi: X \to \mathbb{R}$ there is a set $X_0 \subset X$ containing at most a countable number of points such that $x_0 \notin X_0$ and $\varphi(x) = \varphi(x_0)$ at $x \in X \setminus X_0$.

Or, in another words:

(2_0) For an arbitrary continuous function $\varphi: X \to \mathbb{R}$ a set X_0 of all points $x \in X$ such that $\varphi(x) \neq \varphi(x_0)$, has no more than a countable number of points. At the same time, it is clear that $x_0 \notin X_0$.

Let $X = A(\mathfrak{m})$, $Y = A(\mathfrak{n})$, where $\aleph_0 < \mathfrak{m} < \mathfrak{n}$. Let x_0 and y_0 be accumulation points, respectively, of spaces X and Y. Let us put $Z = X \times Y \setminus \{(x_0, y_0)\}$.

For $x \in X \setminus \{x_0\}$ let us define a set

$$Y_0(x) = \{y \in Y: f(x, y) \neq f(x, y_0)\} \subset Y \setminus \{y_0\}$$

and put

$$Y_0 = \bigcup_{x \in X \setminus \{x_0\}} Y_0(x).$$

It is clear that $Y_0 \subset Y \setminus \{y_0\}$.

For every $x \in X \setminus \{x_0\}$, and for every $y \in Y \setminus Y_0$ the following equality holds

$$f(x, y) = f(x, y_0). \tag{1}$$

By virtue of the property (2_0) for a subset

$Y_0(x) \cong \{(x, y) \in \{x\} \times Y: f(x, y) \neq f(x, y_0)\} \subset \{x\} \times Y$

we have $|Y_0(x)| \leq \aleph_0$. Therefore, $|Y_0| \leq \mathfrak{m}$.

Now, choose an arbitrary $\bar{y} \in Y \setminus (Y_0 \cup \{y_0\})$ and define a set

$$X_0 = \{x \in X : f(x, \bar{y}) \neq f(x_0, \bar{y})\} \subset X \setminus \{x_0\}. \quad (2)$$

Again (2_0) implies that $|X_0| \leq \aleph_0$.

Put

$$Z_0 = (X_0 \times Y) \cup (X \times Y_0).$$

Let $r = f(x_0, \bar{y})$. By virtue of (1) and (2), for any point $(x, y) \in Z \setminus Z_0$, such that $x \neq x_0$, we have

$$f(x, y) \underset{(1)}{=} f(x, y_0) \underset{(1)}{=} f(x, \bar{y}) \underset{(2)}{=} f(x_0, \bar{y}) = r.$$

A set $(Z \setminus Z_0) \setminus \left(\{x_0\} \times (Y \setminus \{y_0\})\right)$ is everywhere dense in a space $Z \setminus Z_0$. Consequently, from $(x_0, y) \in Z \setminus Z_0$ it follows that $f(x_0, y) = r$.

Thus, the following property is proved.

(3_0) For each continuous function $f: Z \to \mathbb{R}$ there exists such a real number r, such sets $X_0 \subset X \setminus \{x_0\}$, $Y_0 \subset Y \setminus \{x_0\}$, with $|X_0| \leq \aleph_0$, $|Y_0| \leq \mathfrak{m}$ and

$$f(x, y) = r$$

at $(x, y) \in Z \setminus Z_0$.

An example of a regular space on which every continuous function is constant

Let $X = A(\mathfrak{m})$, $Y = A(\mathfrak{n})$, where $\aleph_0 < \mathfrak{m} < \mathfrak{n}$. Let x_0 and y_0 be accumulation points, respectively, of the spaces X and Y. Let us put

$$Z = X \times Y \setminus \{(x_0, y_0)\}.$$

For each positive integer i we define sets

$$Z_i = Z \times \{i\} \text{ and } Z_{-i} = Z \times \{-i\}.$$

Let

$$Z^{**} = (\oplus_{i=1}^{\infty} Z_i) \cup (\oplus_{i=1}^{\infty} Z_{-i}).$$

Take elements $z,\ z' \notin Z^{**}$, $z \neq z'$, and we introduce a topology on the set $H^* = Z^{**} \cup \{z, z'\}$ using the neighborhood system $\{\mathcal{B}(x)\}_{x\in H^*}$, where for any $x \in Z^{**}$ a collection $\mathcal{B}(x)$ is a family of all open subsets in Z^{**} containing x,

$$\mathcal{B}(z) = \{U_i(z)\}_{i=1}^{\infty}, \text{ where } U_i(z) = H^* \setminus \left(\bigoplus_{i=1}^{\infty} Z_{-i} \cup \{z'\} \cup \bigcup_{j=1}^{i} Z_j\right)$$

and

$$\mathcal{B}(z') = \{U_i(z')\}_{i=1}^{\infty}, \text{ where } U_i(z') = H^* \setminus \left(\bigoplus_{i=1}^{\infty} Z_i \cup \{z\} \cup \bigcup_{j=1}^{i} Z_{-j}\right).$$

The resulting space H^* is a completely regular space. It is clear that Z^{**} is a subspace of H^*. Let us define the equivalence relation R on H^*, the equivalence classes according to which have the form

$\{(x, y_0, -i-1), (x, y_0, -i), (x, y_0, i), (x, y_0, i+1)\}$ for $x \in X \setminus \{x_0\}$ and odd i,

$\{(x_0, y, -i-1), (x_0, y, -i), (x_0, y, i), (x_0, y, i+1)\}$ for $y \in X \setminus \{y_0\}$ and even i,

$\{(x, y, i)\}$ for $x \in X \setminus \{x_0\}$, $y \in X \setminus \{y_0\}$ and every i,

$\{z\}$ and $\{z'\}$. Therefore, the quotient space $H = H^*/R$ is obtained by identifying the corresponding points in $A \times \{i\}$, $A \times \{i+1\}$, $A \times \{-i\}$ and $A \times \{-i-1\}$ for each odd i and identifying the corresponding points in $B \times \{i\}$, $B \times \{i+1\}$, $B \times \{-i\}$ and $B \times \{-i-1\}$ for each even i. Here as above

$$A = \{(x, y_0):\ x \in X \setminus \{x_0\}\} \subset Z,$$

$$B = \{(x_0, y):\ y \in Y \setminus \{y_0\}\} \subset Z.$$

It is clear that

$$z,\ z' \notin A \times \{i\} \text{ and } z,\ z' \notin A \times \{-i\} \text{ for every odd } i,$$

$$z,\ z' \notin B \times \{i\} \text{ and } z,\ z' \notin B \times \{-i\} \text{ for every even } i.$$

Full preimages of points under natural mapping $q: H^* \to H$ are one-point or four-point sets. Therefore, each point of the space H (that is, each equivalence class) forms a closed set, which means the space H is a T_1-space. Moreover, the space H is a T_3-space. Consequently, the space H is regular.

Take points $t = q(z)$, $t' = q(z')$ and closed sets $F = q(A \times \{1\})$ and $F' = q(A \times \{-1\})$. By the construction of the equivalent relation R, we have $F' = F$. It is clear that $t \notin F$ and $t' \notin F$. Now it remains to note that for each continuous function $f: T \to [0,1]$, such that $f(F) = \{r\}$ it occurs $f(t) = f(t') = r$.

Now let S be an arbitrary regular space and H be the above defined space. We provide the product $Y = S \times H$ with the topology generated with the topology of neighborhoods:

of the view $O(s,h) = \{s\} \times V$ for points $(s,h) \in S \times H$, $h \neq t$, where $V \subset H \setminus \{t\}$ are all possible open sets such that $h \in V$;

of the view $O(s,t) = \bigcup_{s' \in U} (\{s'\} \times V_{s'})$ for point $(s,t) \in S \times H$, where U is a neighborhood of the point s in the space S, a $V_{s'}$ is a neighborhood of the point t in the space H.

Since S is closed in S and $\{t'\}$ is closed in H, the set $S \times \{t'\}$ is closed in the space $Y = S \times H$. We identify $S \times \{t'\}$ of the space Y to a point. Then every continuous function $f: H(S) \to R$ is constant on a regular space S.

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